# k-core percolation on multiplex networks

N. Azimi-Tafreshi,[1] J. Gómez-Gardeñes,[2,3] and S. N. Dorogovtsev[4,5]
[1]*Physics Department, Institute for Advanced Studies in Basic Sciences, 45195-1159 Zanjan, Iran*
[2]*Departamento de Física de la Materia Condensada, Universidad de Zaragoza, E-50009 Zaragoza, Spain*
[3]*Instituto de Biocomputación y Física de los Sistemas Complejos (BIFI), Universidad de Zaragoza, E-50018 Zaragoza, Spain*
[4]*Departamento de Física da Universidade de Aveiro & I3N, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal*
[5]*A. F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia*
(Received 10 May 2014; published 29 September 2014)

We generalize the theory of $k$-core percolation on complex networks to $\mathbf{k}$-core percolation on multiplex networks, where $\mathbf{k} \equiv (k_1, k_2, \ldots, k_M)$. Multiplex networks can be defined as networks with vertices of one kind but $M$ different types of edges, representing different types of interactions. For such networks, the $\mathbf{k}$-core is defined as the largest subgraph in which each vertex has at least $k_i$ edges of each type, $i = 1,2,\ldots,M$. We derive self-consistency equations to obtain the birth points of the $\mathbf{k}$-cores and their relative sizes for uncorrelated multiplex networks with an arbitrary degree distribution. To clarify our general results, we consider in detail multiplex networks with edges of two types and solve the equations in the particular case of Erdős-Rényi and scale-free multiplex networks. We find hybrid phase transitions at the emergence points of $\mathbf{k}$-cores except the $(1,1)$-core for which the transition is continuous. We apply the $\mathbf{k}$-core decomposition algorithm to air-transportation multiplex networks, composed of two layers, and obtain the size of $(k_1,k_2)$-cores.

DOI: 10.1103/PhysRevE.90.032816 PACS number(s): 64.60.aq, 89.75.Fb, 05.70.Fh, 64.60.ah

## I. INTRODUCTION

In the past decade, during the advent of network science, a number of statistical descriptions were proposed to characterize the structure of the interactions of many diverse complex systems [1–4]. One of fundamental features characterizing the structure of an infinite network is the size of its giant connected component, i.e., the size of the largest connected cluster in the network, containing a finite fraction of vertices. The existence of a giant connected component is particularly important in networks carrying flows of different natures, such as viruses and rumors in social systems, data in technological networks, or goods and humans in transportation systems. Thus, the maximum capacity for the spread and transport in such systems is limited by the size of the giant connected component. Apart from this practical meaning, the giant connected component is the relevant order parameter that shows the formation of a macroscopic cluster in the context of ordinary percolation [5–7].

Apart from ordinary percolation, which results in a continuous phase transition for the emergence of a giant connected component (i.e., containing a finite fraction of vertices) in an infinite network, a number of generalizations of percolation were introduced. These versions lead to other kinds of giant connected components, whose emergence is associated with different phase transitions [8–13]. Among these generalizations, there is $k$-core percolation, in which a giant $k$-core exists if the vertex mean degree of the network exceeds some threshold [8,9]. The $k$-core of a network is defined as the largest subgraph whose vertices have degree at least $k$. The phase transition associated with the $k$-core problem is not continuous if $k \geqslant 3$. In this paper, as is typical for percolation problems, we discuss only infinite networks. If such a network contains finite length loops (mathematicians say "cycles"), a $k$-core can consist of separate parts. Analytical calculations, however, are practically possible only for infinite networks with only infinite loops (so-called locally treelike networks). Our calculations will be only for networks of this kind and their generalizations. In these networks, a finite neighborhood of a vertex is a tree in the sense that it has no loops. One can see that proper trees cannot have $(k \geqslant 2)$-cores, so finite $k$-cores are absent in locally treelike networks. Hence, the $k$-core in a network of this kind (if it exists) consists only of a single giant component, i.e., in these networks, the $k$-core is always giant. This situation differs sharply from ordinary percolation, in which finite connected components are present even in locally treelike networks. Thus, each two vertices in the giant $k$-core are interconnected by at least $k$ paths, which may partially overlap with each other.

The $k$-core of a given graph can be obtained by a recursive pruning algorithm that, at each step, removes all existing vertices with degrees less than $k$. As the result of this pruning, the network is decomposed into a hierarchical set of progressively enclosed $k$-cores with the highest $k$-core being placed in the center. The application of this graph decomposition technique to large real-world networks makes it possible to describe qualitatively their structure in terms of the complete set of their $k$-cores [14,15]. Moreover, it was shown that the most efficient vertices in spreading processes are those belonging to higher $k$-cores of a network [16]. For these reasons, during the last years the $k$-core organization of complex networks has been extensively studied [8,9,17]. The most remarkable result is that for $k \geqslant 3$ a $k$-core emerges discontinuously at the percolation threshold through a hybrid phase transition, combining a discontinuity and a critical singularity and thus breaking the usual continuous scenario of ordinary percolation.

Very recently, it has been considered that most of real-world networks are not isolated objects but composed of many coupled, interdependent networks such that the function of one network depends on the others [18,19]. For such networks, the pruning of vertices in one network can lead to removal of dependent vertices in other networks. It was found that generalized percolation properties of these interdependent networks differ strongly from ordinary percolation on a single network [20–23]. In fact, they more resemble features of $k$-core percolation ($k \geqslant 3$) in single networks including the hybrid phase transition.

The simplest particular case of interdependent networks are multiplex networks in which each vertex depends on at most one vertex of other networks. Multiplex networks can be treated as superpositions of several networks (sometimes called layers) with different edges [24,25]. In other words, all vertices in these networks are of the same type, but the edges are of different types (colors). Note that here a vertex does not necessarily have all types of connections. Multiplex networks have recently attracted a lot of attention as they are the kind of substrates representing better the interaction patterns occurring in many real systems, in which several ways for the interaction between the elements coexist. This is the case of transportation, social, and technological networks among others.

The structural and dynamical properties of multiplex networks were studied in several contexts, including diffusion [26,27], evolutionary games [28], Boolean dynamics [29], epidemics [30], and, of course, percolation [31,32]. In this way, in Ref. [32], a viable component for multiplex networks is defined as a set of vertices in which, for every type of edges, each two vertices are interconnected by at least one path following only edges of this type. Similarly to $k$-cores, in locally treelike multiplex networks, the viable component can be only giant and single. Clearly, the giant viable component of a multiplex network is a subgraph of the giant connected components of all single networks (layers). The theory of ordinary percolation on multiplex networks shows a hybrid transition which is the birth of the giant viable components, similar to what happens in $k$-core percolation in single networks. In Ref. [20] it was shown that the giant viable component is a subgraph of the so-called mutual component (or mutually connected component). By definition, a vertex belongs to the mutual component if for each type of its edges at least one edge has the second end in the mutual component. Hereafter we focus on the giant viable component in locally treelike multiplex networks.

In this paper we study the organization of specific subgraphs, **k**-cores for multiplex networks, where $\mathbf{k} \equiv (k_1,k_2,\ldots,k_M)$. We define the **k**-core of a multiplex network as its largest subgraph in which each vertex has at least $k_i$ edges of each type $i$. We consider in detail locally treelike networks in which a finite neighborhood of a vertex has no loops. Similarly to $k$-cores and viable components, for each given **k**, only a giant, single **k**-core can exist in a multiplex network of this kind. This is why the term "**k**-core" in these networks implies that it is giant. We demonstrate for these networks that if $k_i \geqslant 2$ for all $i$, then between each two vertices in the **k**-core, there are at least $k_i$ paths, following only edges of type $i$. We will calculate parameters of the giant **k**-cores in basic locally treelike multiplex networks.

The paper is organized as follows. In Sec. II, we introduce an algorithm for the **k**-core decomposition of multiplex networks and present an analytical framework enabling us to describe the nature of the transitions corresponding to the emergence of **k**-cores with arbitrary $(k_1,k_2,\ldots,k_M)$. We apply our general results to the Erdős-Rényi and scale-free uncorrelated multiplex networks. In Sec. III, as an application to the real multiplexes, we apply the **k**-core algorithm to air-transportation multiplexes and compare the results with our analytical predictions.

## II. k-CORE OF A MULTIPLEX NETWORK

### A. Analytical framework

Let us consider an uncorrelated multiplex network, having $M$ types of edges, with a given joint degree distribution $P(q_1,q_2,\ldots,q_M)$, which has a locally treelike structure in the infinite network limit. These networks have only infinite loops. It is the presence of infinite loops that makes possible a giant **k**-core. Note the difference from infinite proper trees, in which loops are absent, and viable components and **k**-cores are consequently absent. For simplicity, we also assume that the network is sparse and completely uncorrelated, though, in principle, correlations between different edges, $i = 1,2,\ldots,M$, of a vertex might be easily taken into account. The **k**-core of a multiplex network is defined as its largest subgraph in which each vertex has at least $k_i$ edges of each type $i$. To obtain the **k**-core of a multiplex network, we use the following pruning algorithm: At each step we remove every vertex if for at least one type of edge $i$, $q_i < k_i$. As the result of the pruning, the degrees of some vertices change. If there are still vertices which one can prune, we remove them in the next step. The pruning is continued until no vertex remains of degree $q_i$ less than the threshold $k_i$. Figure 1 shows the **k**-core decomposition for a multiplex network with two types of edges.

To find the size of the giant **k**-core, for each type $i$ of edge, we define $x_i$ as the probability that an end vertex of a randomly chosen edge of type $i$ is the root of an infinite subtree of type $i$. The subtree of type $i$ is, by definition, a tree whose vertices have at least $k_i - 1$ edges of type $i$ and at least $k_j$ edges of each of other types $j \neq i$ edges. Probabilities $x_1$ and $x_2$ for a multiplex network with two types of edges are

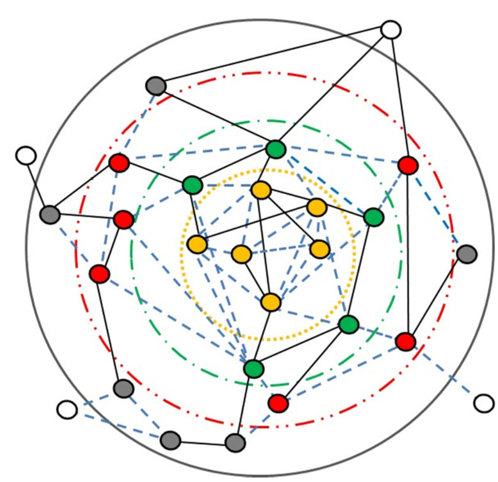

FIG. 1. (Color online) The $(k_1,k_2)$-core decomposition for a multiplex network with two types of edges. Solid black and dashed blue edges are edges of type 1 and type 2, respectively. The cores from the outermost to the innermost are the $(1,1)$-core, the $(1,2)$-core, the $(2,2)$-core, and $(1,3)$-core.

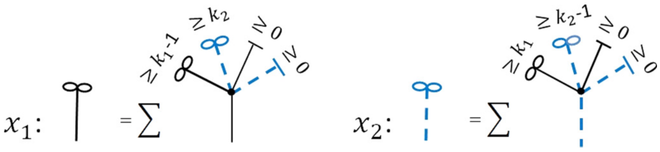


FIG. 2. (Color online) Schematic representation of the self-consistency equations for the probabilities $x_1$ and $x_2$. The solid black and dashed blue lines with infinity symbols at one end represent probabilities $x_1$ and $x_2$, respectively. The edges lead to finite components, namely, $1 - x_1$ and $1 - x_2$, are shown by solid and dashed lines with cuts at one end.

schematically shown in Fig. 2. These probabilities play the role of the order parameters of the phase transition associated with the emergence of the **k**-cores. We can write the self-consistency equations for probabilities $x_i$ using the locally treelike structure of the networks,

$$x_i = \sum_{\mathbf{q}} \frac{q_i P(\mathbf{q})}{\langle q_i \rangle} \left[ \sum_{s=k_i-1}^{q_i-1} \binom{q_i-1}{s} x_i^s (1-x_i)^{q_i-1-s} \right] \times \prod_{\substack{j=1 \\ j \neq i}}^{M} \left[ \sum_{s'=k_j}^{q_j} \binom{q_j}{s'} x_j^{s'} (1-x_j)^{q_j-s'} \right], \quad (1)$$

where $\mathbf{q} \equiv (q_1, q_2, \ldots, q_M)$ is the degree of a vertex.

Let us briefly explain Eq. (1). The probability that the end vertex of a randomly chosen edge of type $i$ has degree $\mathbf{q}$ is $q_i P(\mathbf{q})/\langle q_i \rangle$. The combinatorial multiplier $\binom{m}{n}$ gives the number of ways one can choose $n$ edges from a sample of $m$ edges. At least $k_i - 1$ edges of $q_i - 1$ edges of type $i$ (other edges than the starting one) must lead to an infinite subtree of type $i$ (probability $x_i$) and at least $k_j$ edges of each of $q_j$ edges ($j \neq i$) must lead to the infinite subtrees of type $j$ (probability $x_j$).

Using these probabilities, we can obtain the relative size of the giant **k**-core. A vertex is in the giant **k**-core when, for each type of edge $i$, the vertex has at least $k_i$ edges, leading to infinite $i$ subtrees. The probability that a vertex belongs to the $(k_1,k_2)$-core for multiplex networks with edges of two types, is shown schematically in Fig. 3. Hence, the relative size of the core is given by the following expression:

$$n_{\mathbf{k}\text{-core}} = \sum_{\mathbf{q}} P(\mathbf{q}) \prod_{i=1}^{M} \left[ \sum_{s=k_i}^{q_i} \binom{q_i}{s} x_i^s (1-x_i)^{q_i-s} \right]. \quad (2)$$

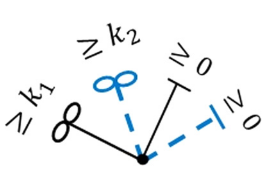


FIG. 3. (Color online) Schematic representation of the probability that a vertex belongs to $(k_1,k_2)$-core, which is the relative size $n_{\mathbf{k}\text{-core}}$ of the **k**-core; see Eq. (2). The figure shows the terms contributing to this probability.

We can now rewrite Eqs. (1) and (2) using generating functions [7], which enable us to solve these equations analytically. For a single network with a given degree distribution $P(q)$, the generating function $G(x)$ is defined as

$$G(x) \equiv \sum_q P(q) x^q. \quad (3)$$

If we assume that there is no correlation between the degrees $q_1, q_2, \ldots, q_M$, so that $P(\mathbf{q}) = P(q_1)P(q_2), \ldots, P(q_M)$, then Eqs. (1) and (2) can be rewritten as

$$x_i = \left[ 1 - \frac{1}{\langle q_i \rangle} \sum_{s=0}^{k_i-2} x_i^s \frac{G_i^{(s+1)}(1-x_i)}{s!} \right] \times \prod_{\substack{j=1 \\ j \neq i}}^{M} \left[ 1 - \sum_{s'=0}^{k_j-1} x_j^{s'} \frac{G_j^{(s')}(1-x_j)}{s'!} \right] \quad (4)$$

and

$$n_{\mathbf{k}\text{-core}} = \prod_{i=1}^{M} \left[ 1 - \sum_{s=0}^{k_i-1} x_i^s \frac{G_i^{(s)}(1-x_i)}{s!} \right], \quad (5)$$

where we used the notation $G^s(x)$ for the $s$th derivatives of $G(x)$, which produce the higher moments of the distribution $P(q)$. Note that in these equations, $(1 - x)$ is the argument of the generation function and its derivatives.

In general, Eq. (4) is a set of self-consistency equations of the form $x_i = f_i(x_1, x_2, \ldots, x_M)$. Let us choose $\langle q_i \rangle$ as control parameters. For small values of $\langle q_i \rangle$, there is only zero root of these equations and, hence, the giant **k**-core does not exist. The critical value of a control parameter is obtained from the system of equations $x_i = f_i(x_1, x_2, \ldots, x_M)$ together with the condition $\det[\mathbf{J} - \mathbf{I}] = 0$ for the Jacobian matrix $\mathbf{J}$, defined as $J_{ij} = \partial f_j / \partial x_i$, and $\mathbf{I}$ is the identity matrix. Figure 4 demonstrates that in the symmetric case of the (2,2)-core of a multiplex network with identical degree distributions for two

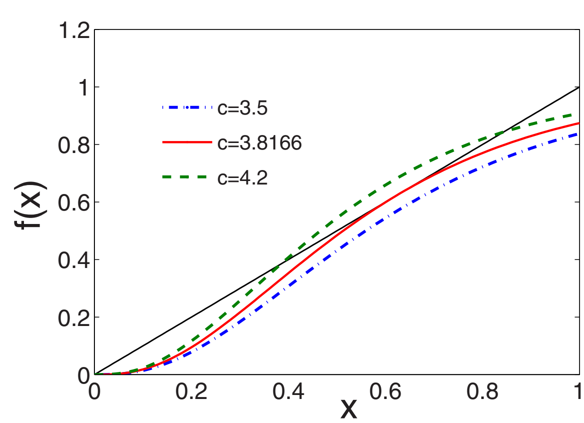


FIG. 4. (Color online) Graphical solution of Eq. (15) for the (2,2)-core in a multiplex graph with a Poisson degree distribution. The straight line and the curves $f(x)$ show, respectively, the left- and right-hand sides of the equation as functions of $x$ for different values of the mean degree $c$. The solution appears above the critical value $c = 3.8166$, at which the right-hand side curve $f(x)$ starts to intersect the straight line. The physical solution is provided by the largest root of the equation $x = f(x)$ (the upper intersection in the plot).

types of edges, the critical point occurs when $f(x)$ starts to meet $x$ at $x > 0$. The values of the jumps of $x_i$ at the critical point follow from this system of equations. The physical solution above the critical point is given by the largest root of the equations $x_i = f_i(x_1,x_2,\dots,x_M)$ by the following reason: With increasing the control parameters, the size of **k**-core should increase. One can see that a hybrid phase transition combining a jump and a square-root critical singularity takes place at the critical point. Expanding $f_i(x_1,x_2,\dots,x_M)$ around the transition point enables us to find the singularity of $x_i$ and hence $n_{\mathbf{k}\text{-core}}$. One can find that this is a square-root singularity. For example, in a simple symmetric case of the (2,2)-cores in a multiplex network in which the degree distributions for the two types of edges coincide and so $\langle q_1\rangle = \langle q_2\rangle \equiv c$, we have a single order parameter $x$, which is the largest root of the equation $x = f(x)$; see Fig. 4. Expanding $f(x)$ near the critical point, $c = c_c + \delta$, $x = x_c + \Delta$, we have

$$x_c + \Delta = f(x_c,c_c) + \partial_x f(x,c)|_{x=x_c,c=c_c}\Delta + \partial_c f(x,c)|_{x=x_c,c=c_c}\delta + \tfrac{1}{2}\partial^2_{xx} f(x,c)|_{x=x_c,c=c_c}\Delta^2 + \cdots . \quad (6)$$

Taking into account $x_c = f(x_c,c_c)$ and $1 = \partial_x f(x,c)|_{x=x_c,c=c_c}$, we obtain the square-root singularity for the order parameter:

$$\Delta \cong \left[-2\frac{\partial_c f(x_c,c_c)}{\partial^2_{xx} f(x_c,c_c)}\right]^{1/2}\sqrt{\delta}. \quad (7)$$

A similar square-root singularity can be found for the size of the **k**-core.

The structure of the **k**-core is described by the degree distribution $P_{\mathbf{k}}(\mathbf{q})$ defined as the probability to find a vertex of degree $\mathbf{q} \equiv (q_1,q_2,\dots,q_M)$ in the **k**-core,

$$P_{\mathbf{k}}(\mathbf{q}) = \frac{n_{\mathbf{k}}(\mathbf{q})}{n_{\mathbf{k}\text{-core}}}, \quad (8)$$

where $n_{\mathbf{k}}(\mathbf{q})$ is the fraction of vertices with degree $\mathbf{q}$, which fall into the **k**-core. This fraction is given by the following expression:

$$n_{\mathbf{k}}(\mathbf{q}) = \sum_{\mathbf{q}'\geqslant\mathbf{q}} P(\mathbf{q}')\prod_{i=1}^{M}\left[\binom{q_i'}{q_i} x_i^{q_i}(1-x_i)^{q_i'-q_i}\right]. \quad (9)$$

In particular, Eq. (9) helps us to find the size of the corona (a subset of vertices of degree **k** in the **k**-core, which generalizes the notion of the corona clusters of the $k$-core). Setting $\mathbf{q} = \mathbf{k}$ and making use of generating functions, we find the relative size of the corona:

$$n_{\mathbf{k}}(\mathbf{k}) = \prod_{i=1}^{M} x_i^{k_i}\frac{G_i^{(k_i)}(1-x_i)}{k_i!}. \quad (10)$$

Furthermore, the average total degree $q_1 + q_1 + \cdots + q_M$ of a vertex in the **k**-core can be obtained using the degree distribution of the **k**-core in the following way:

$$c_{\mathbf{k}\text{-core}} = \sum_{\mathbf{q}}(q_1 + q_2 + \cdots + q_M)P_{\mathbf{k}}(\mathbf{q}). \quad (11)$$

To clarify our results, in the following we consider Erdős-Rényi and scale-free multiplex networks with two types of edges to describe the **k**-core organization of these multiplex networks.

### B. Erdős-Rényi networks

Let us first consider Erdős-Rényi multiplex networks with Poisson degree distributions: $P(q_1) = c_1^{q_1}e^{-c_1q_1}/q_1!$, $P(q_2) = c_2^{q_2}e^{-c_2q_2}/q_2!$, where $c_1$ and $c_2$ are the mean vertex degrees for types 1 and 2 edges, respectively. For the Poisson distribution, the generating function and its $s$th derivative are $G_i(x) = e^{-c_i(1-x)}$ and $G_i^s(x) = c_i^s e^{-c_i(1-x)}$, respectively.

For the sake of simplicity, let us consider the symmetric case $c_1 = c_2 \equiv c$. The largest core is the core with $k_1 = k_2 = 1$, that is, (1,1)-core. In this case one can obtain $x_1 = x_2 \equiv X$, such that $X = 1 - e^{-cX}$. For $c > 1$, this equation has a nonzero nontrivial solution. Figure 5(a) shows the relative size of the (1,1)-core, displaying a continuous transition at $c = 1$, and is

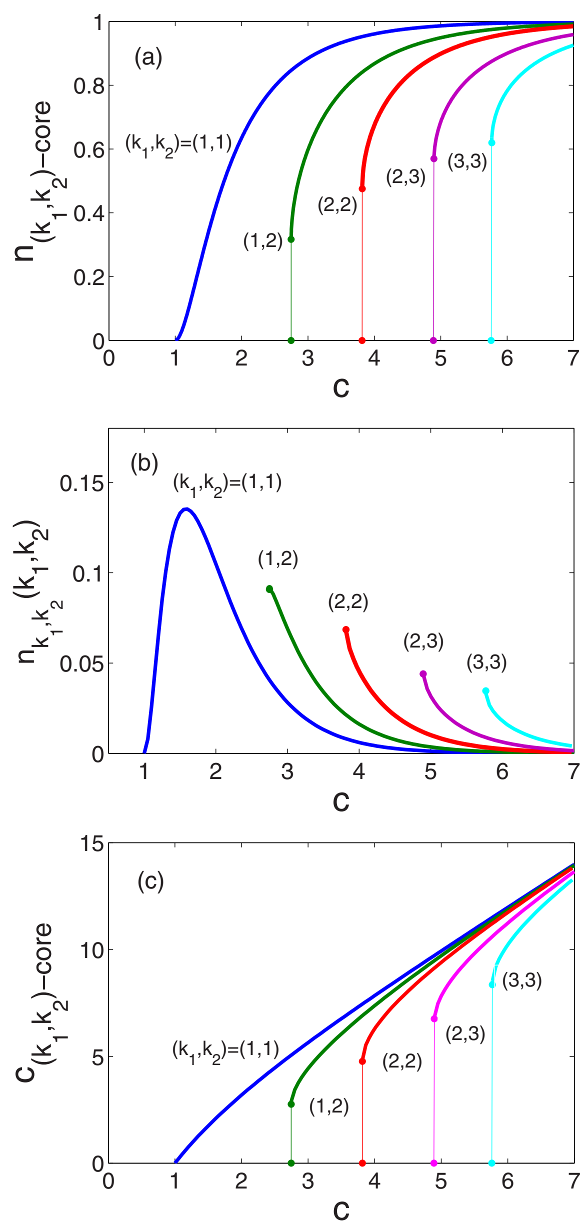


FIG. 5. (Color online) The relative sizes of (a) **k**-core and (b) corona, and (c) the mean vertex degree of a **k**-core in Erdős-Rényi multiplex networks for some values of $k_1$ and $k_2$ vs the vertex mean degree $c_1 = c_2 = c$ of the network.

given by the expression

$$n_{(1,1)\text{-core}} = (1 - e^{-cX})^2 \cong 4(c-1)^2, \tag{12}$$

Note that there exist paths between the vertices of the $(1,1)$-core but these paths do not necessarily follow edges of the same type. Hence, the giant $(1,1)$-core does not coincide with the giant viable component. In ordinary single networks, the giant 1-core also does not coincide with the giant connected component.

Furthermore, if only one of the two $k_i$ values is equal to 1, then in the $(k_1,k_2)$-core there are less than $k_1 + k_2$ paths between two vertices following edges of the corresponding types. However, in this case there are $k_1 + k_2 \geqslant 3$ paths between each two vertices within the $(k_1,k_2)$-core following edges of alternating types. So such a **k**-core is not a subgraph of the viable component.

For instance, let us consider the $(1,2)$-core. In this case, one can find the following equations for $x_1$ and $x_2$:

$$\begin{aligned} x_1 &= (1 - e^{-c_2x_2} - c_2x_2e^{-c_2x_2}), \\ x_2 &= (1 - e^{-c_2x_2})(1 - e^{-c_1x_1}). \end{aligned} \tag{13}$$

Using these probabilities, the relative size of the $(1,2)$-core is given by the following expression:

$$n_{(1,2)\text{-core}} = (1 - e^{-c_1x_1})(1 - e^{-c_2x_2} - c_2x_2e^{-c_2x_2}). \tag{14}$$

Figure 5(a), in particular, shows the relative size of the $n_{(1,2)\text{-core}}$ in the symmetric case of $c_1 = c_2 = c$. While the giant core is nonviable, a hybrid transition appears at the emergence point of the core at $c \simeq 2.7461$.

Let us now assume that each $k_i$ exceeds 1. In this case, considering the tree for $x_i$ in Fig. 2, one can show that for every type $i$ of edges, each two vertices within this core are interconnected by at least $k_i$ paths following only edges of this type (see Fig. 3). Hence, **k**-core is a subgraph of the viable component if all the components $k_i$ of the vector **k** exceed 1.

As an example, let us consider $k_1 = k_2 = 2$. In the symmetric case of $c_1 = c_2 \equiv c$, we find $x \equiv x_1 = x_2$ satisfying the following equation:

$$x = (1 - e^{-cx})(1 - e^{-cx} - cxe^{-cx}). \tag{15}$$

Solving the equation for $x$, we can find that the transition occurs at $c \simeq 3.8166$ (see Fig. 4). The relative size of $(2,2)$-core is given by the expression

$$n_{(2,2)\text{-core}} = (1 - e^{-cx} - cxe^{-cx})^2. \tag{16}$$

This core emerges discontinuously at the transition point, as shown in Fig. 5(a). Figure 5(a) also shows the relative sizes of the $(k_1,k_2)$-cores for some other values of $k_1$ and $k_2$. There is a jump at the birth points of the cores, which, together with a critical singularity, points out a hybrid transition for $k_1 \geqslant 2$ and $k_2 \geqslant 2$.

One can obtain the relative size of the corona for the Erdős-Rényi networks using Eq. (10) as

$$n_{k_1,k_2}(k_1,k_2) = \frac{(c_1x_1)^{k_1}e^{-c_1x_1}}{k_1!}\,\frac{(c_2x_2)^{k_2}e^{-c_2x_2}}{k_2!}. \tag{17}$$

Figure 5(b) displays the dependence of the corona sizes on the vertex mean degree $c$. Furthermore, using Eq. (8), we can write the degree distribution of the $(k_1,k_2)$-core, that is,

$$P_{(k_1,k_2)}(q_1,q_2) = \frac{1}{n_{(k_1,k_2)\text{-core}}}\,\frac{\left(x_1^{q_1}c_1^{q_1}e^{-c_1x_1}\right)\left(x_2^{q_2}c_2^{q_2}e^{-c_2x_2}\right)}{q_1!q_2!}. \tag{18}$$

Hence, the total vertex mean degree of the $(k_1,k_2)$-core for uncorrelated Erdős-Rényi networks with two types of edges, 1 and 2, is

$$c_{(k_1,k_2)} = c_1x_1 + c_2x_2. \tag{19}$$

As one can notice from Fig. 5(c), in the symmetric case of $c_1 = c_2 = c$, the total mean degree of the $(k_1,k_2)$-core changes almost linearly with $c$.

To round off the study of Erdős-Rényi multiplexes, we compare **k**-core percolation on a Erdős-Rényi multiplex network and $k$-core percolation on its counterpart single network. As one can see in Fig. 6(a), the $(3,3)$-core emerges at a much higher mean degree value than the 3-core of the corresponding single Erdős-Rényi graph. In general, $(k_1,k_2)$-core percolation on the multiplex network has a higher threshold than the $k_1$ or $k_2$-core on single networks. However, the $(k_1+k_2)$-core in a single network has a higher threshold than the $(k_1,k_2)$-core in the corresponding multiplex networks. Figure 6(b) compares the relative sizes of the $(2,2)$-core of the Erdős-Rényi multiplex network with the 4-core in the corresponding ordinary single Erdős-Rényi networks.

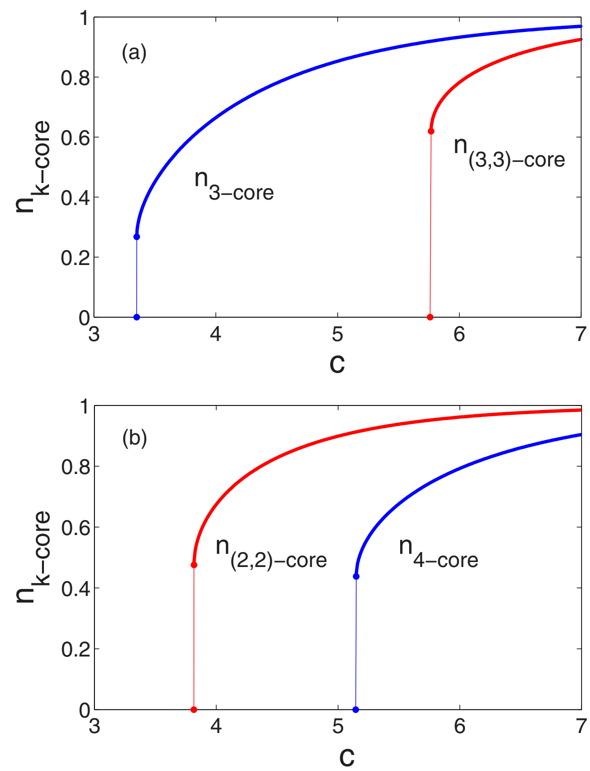


FIG. 6. (Color online) (a) The relative sizes and the birth points of the $(3,3)$-cores in the Erdős-Rényi multiplex network are compared with the 3-core in the corresponding ordinary single Erdős-Rényi networks. (b) The relative size of the $(2,2)$-core of the Erdős-Rényi multiplex network vs the mean degree of its vertices, compared with the 4-core in the corresponding ordinary single Erdős-Rényi networks.

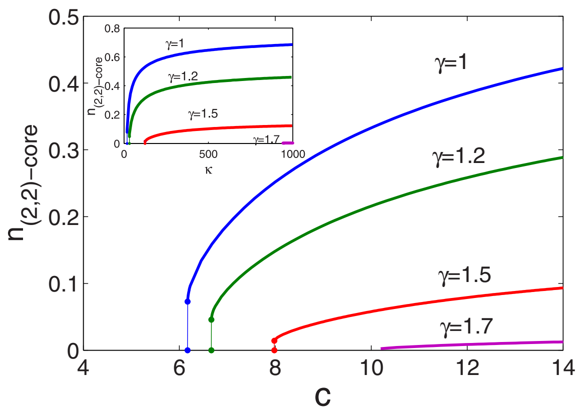


FIG. 7. (Color online) The relative size of the (2,2)-core for scale-free multiplex networks with an exponential degree cutoff $\kappa$ at different values of the exponent $\gamma$ of the degree distribution. The core disappears as $\gamma$ increases and approaches 2. The inset shows the dependence of the relative size of the (2,2)-core on the cutoff parameter $\kappa$ for different values of the exponent $\gamma$.

### C. Scale-free networks

Let us consider scale-free multiplex networks with two types of edges. For simplicity we assume that the degrees $q_1$ and $q_2$ are distributed in the same manner with the power-law exponents $\gamma_1 = \gamma_2 \equiv \gamma$ and the mean degrees $\langle q_1 \rangle = \langle q_2 \rangle \equiv c$. First we consider the organization of **k**-cores for the power-law distributed networks with an exponential degree cutoff, i.e., for instance, $P(q) = \frac{q^{-\gamma} e^{-q/\kappa}}{\mathrm{Li}_\gamma(e^{-1/\kappa})}$, where $\mathrm{Li}_n(x)$ is the $n$th polylogarithm of $x$. The mean degree, $c$, is related to the cutoff parameter $\kappa$ as $c = \frac{\mathrm{Li}_{\gamma-1}(e^{-1/\kappa})}{\mathrm{Li}_\gamma(e^{-1/\kappa})}$. For this distribution, the generating function defined by Eq. (3) is $G(x) = \frac{x\mathrm{Li}_\gamma(e^{-1/\kappa})}{\mathrm{Li}_\gamma(e^{-1/\kappa})}$. The results significantly depend on the low-degree part of the degree distribution. The presence of a cutoff enables us to consider even low values of $\gamma$, including $\gamma < 2$. For each value of the exponent $\gamma$, we vary the cutoff $\kappa$ (and thus the mean degree $c$) as the control parameter. The relative sizes of the **k**-cores are obtained by solving Eqs. (4) and (5). Figure 7 shows $n_{(2,2)\text{-core}}$ for different values of $\gamma$. It becomes clear that the size of the core decreases as $\gamma$ approaches to 2. Hence, for scale-free networks with exponential degree cutoff and, particularly, for the case of pure scale-free networks ($\kappa \to \infty$), the $(k_1,k_2)$-core does not exist for $\gamma > 2$.

Next we consider asymptotically scale-free networks generated by the static model with

$$P(q) = \left\{ \left[ \frac{c(\gamma-2)}{2(\gamma-1)} \right]^{\gamma-1} \Gamma\left[ q-\gamma+1, \frac{c(\gamma-2)}{2(\gamma-1)} \right] \right\} \Big/ \Gamma(q+1), \tag{20}$$

where $\Gamma(s)$ is the gamma function and $\Gamma(s,x)$ is the upper incomplete gamma function [33]. This function in the large $q$ limit is asymptotically power law, $P(q) \sim q^{-\gamma}$ for $\gamma > 2$. The generating function is $G(x) = (\gamma-1)E_n[(1-x)\frac{c(\gamma-2)}{2(\gamma-1)}]$, where $E_n(x) = \int_1^\infty dy e^{-xy} y^{-n}$ is the exponential integral.

For different values of $\gamma$, Fig. 8 shows the relative size of the (2,2)-core and the (2,3)-core and their corresponding

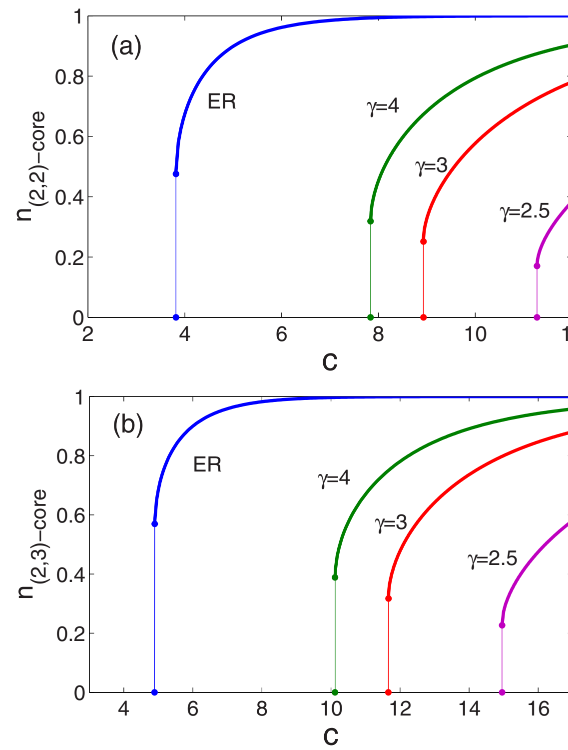


FIG. 8. (Color online) The relative size of (a) (2,2)-core and (b) (2,3)-core, for Erdős-Rényi and asymptotically scale-free networks, with different values of $\gamma$. For scale-free networks with smaller $\gamma$, the critical point shifts to higher mean degree values.

emergence points. The value of the jumps at the critical point increases with increasing $\gamma$. Also, when the exponent $\gamma$ decreases, the transition point moves towards higher values of mean degree. In Fig. 8 we compare the emergence of cores for scale-free and Erdős-Rényi networks. As one can see, the dependency of the cores on $c$ for these networks is similar and, as expected, the curves with larger $\gamma$ approach the result for Erdős-Rényi networks. Note that Figs. 7 and 8 present results obtained for very different distributions. The former is for a power-law distribution with a cutoff, and the latter is for a distribution which approaches a power-law form in the large degree limit.

## III. REAL MULTIPLEX NETWORKS

As an example of real multiplex networks, we consider air-transportation networks, in which vertices represent airports and edges direct flight connections between airports. Since flights can be operated by different airlines, a detailed representation of this kind of systems is given by multiplexes [34]. For the sake of simplicity we consider air-transportation multiplexes composed of two types of edges (two different airlines). Among the possible choices for composing these multiplexes, we have considered both low-cost airlines and major carriers. The main difference between the organization of these airlines is that low-cost companies diversify their main airports and their goal is mainly driven by the economic growth. Instead, major carriers are typically associated with

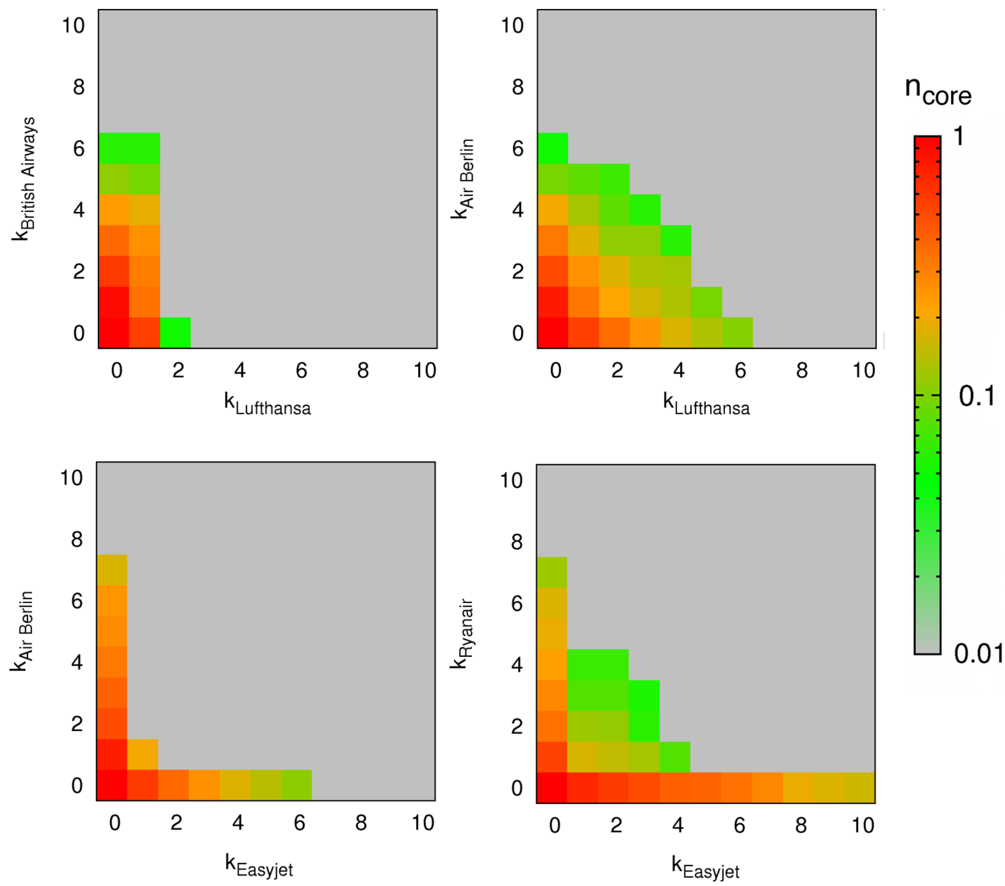


FIG. 9. (Color online) The relative size of the giant **k**-cores of BritishAirway-Lufthansa, AirBerlin-Lufthansa, AirBerlin-EasyJet, and Ryanair-EasyJet multiplex transportation networks, for different values of $k_1$ and $k_2$.

one country as they are originally designed to serve the national and international mobility of the corresponding citizens. These major airlines are thus designed following the so-called *hub-and-spoke* structure in which one airport (the hub) is surrounded by many low-degree vertices forming a kind of starlike graph.

We have considered three different types of multiplexes comprising (i) two low-cost airlines, (ii) two major airlines, and (iii) one low-cost and one major airline. In particular, we have considered these combinations: *EasyJet-Ryanair* and *EasyJet-AirBerlin* (combination of two low-cost airlines), *Lufthansa-British Airways* (combination of two major airlines) and *Lufthansa-AirBerlin* (combination of low-cost and major airlines).

Figure 9 shows the sizes of the giant **k**-cores of these networks for different values of $k_1$ and $k_2$. For the case EasyJet-Ryanair and Lufthansa-AirBerlin (which are two airlines operating in the same country), we obtain more central cores, since the combined companies have many common vertices with a large number of connections for each type of edge. On the other hand, for EasyJet-AirBerlin, the common vertices have a few connections for each type of edge, which are removed in the first steps of **k**-core algorithm. Hence, this multiplex network has only a few cores. Similarly, the Lufthansa-British Airways multiplex network has a few cores, since these two major airlines operate from different countries and thus the connectivity of the overlapping vertices is very different.

In Fig. 10, we have compared the size of the giant (2,2)-core and the giant (2,3)-core for air-transportation multiplex networks with the corresponding analytical results, obtained from Eqs. (4) and (5) in which we made use of the degree distributions of the empirical multiplex networks. As one can see, in some cases there is a noticeable difference between theory and reality, which arises from clustering, degree-degree correlations, and structural motifs in real-world networks, which we did not take into account. Furthermore, the effects of overlapping between edges from different layers in the real-world multiplex networks can also be significant as noted in [35,36].

## IV. CONCLUSION

In this work we have generalized the theory of **k**-core percolation to the multiplex networks. We proposed a pruning algorithm for the **k**-core decomposition of multiplex networks that may be useful for describing the topological structure of these networks. We have analytically solved the **k**-core percolation problem for uncorrelated multiplex networks with arbitrary degree distributions. In the particular case of $(1,1)$-core, the transition is continuous. This does not contradict previous

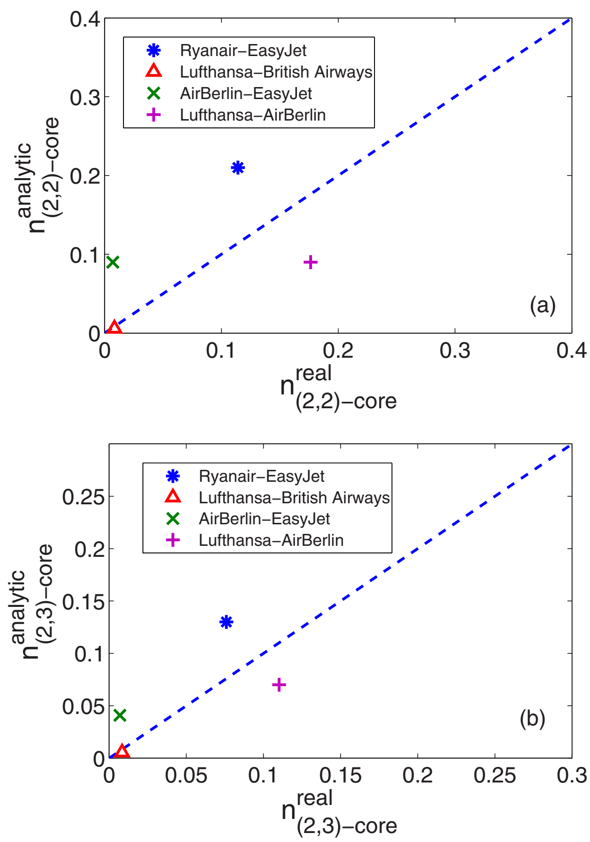


FIG. 10. (Color online) Comparison between **k**-core sizes obtained from empirical data and our theory for the relative sizes of (a) the giant (2,2)-core and (b) the giant (2,3)-core of transportation multiplex networks. Theoretical values are calculated for uncorrelated counterparts of the corresponding empirical multiplex networks.

results for ordinary percolation on multiplex networks, because in this case the (1,1)-core is not a viable component. We showed, however, that the transitions for higher cores are hybrid. In particular, the (1,2) and (2,1)-cores display hybrid transitions. Moreover, we found that the $(k_1,k_2,\ldots,k_M)$-core on uncorrelated multiplex networks has a higher threshold than the $k_i$-core on any counterpart single network $i$. Hence, we conclude that multiplex networks are less robust compared to their counterpart single networks, if we analyze the destruction of the **k**-cores induced by random removal of vertices.

In summary, the **k**-core problem for the multiplex networks turns out to be essentially richer than the $k$-core problem for single networks. The pruning algorithm allows one to extract the **k**-cores in the multiplex networks in an easier way than their viable components. So the **k**-core decomposition of these complex networks is algorithmically efficient. In the analytical framework presented here, we focused on uncorrelated and locally treelike multiplex networks, ignoring the overlap of different types of edges, clustering, and correlations. Our empirical data analysis has revealed that these features may be significant for the sizes and organization of **k**-cores. We suggest that our theory could be extended to consider the case of complex multiplex networks with diverse structural correlations.

## ACKNOWLEDGMENTS

This work was partially supported by the Portuguese FCT under Projects No.PTDC/MAT/114515/2009 and No. PEst-C/CTM/LA0025/2011; the Spanish MINECO under Projects No. FIS2011-25167 and No. FIS2012-38266-C02-01; and the FET IP Projects MULTIPLEX (Grant No. 317532) and PLEXMATH (Grant No. 317614). J.G.G. is supported by the Spanish MINECO through the Ramon y Cajal program.


[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[3] M. E. J. Newman, *Networks: An Introduction* (Oxford University Press, USA, 2010).
[4] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[5] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008).
[6] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford University Press, Oxford, U.K., 2003); S. N. Dorogovtsev, *Lectures on Complex Networks* (Oxford University Press, Oxford, U.K., 2010).
[7] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[8] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett. **96**, 040601 (2006).
[9] A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, Phys. Rev. E **73**, 056101 (2006).
[10] G. Palla, I. Dernyi, I. Farkas, and T. Vicsek, Nature (London) **435**, 814 (2005).
[11] I. Derenyi, G. Palla, and T. Vicsek, Phys. Rev. Lett. **94**, 160202 (2005).
[12] D. Achlioptas, R. M. D'Souza, and J. Spencer, Science (New York, N. Y.) **323**, 1453 (2009).
[13] R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett. **105**, 255701 (2010).
[14] I. Alvarez-Hamelin, L. Dell-Asta, A. Barrat, and A. Vespignani, in *Advances in Neural Information Processing Systems*, edited by Y. Weiss, B. Schölkopf, and J. Platt, Vol. 18 (MIT Press, Cambridge, MA, 2006), p. 41.
[15] S. Carmi, S. Havlin, S. Kirkpatrick, Y. Shavitt, and E. Shir, PNAS **104**, 11150 (2007).
[16] M. Kitsak, L. K. Gallos, S. Havlin, F. Liljeros, L. Muchnik, H. E. Stanley, and H. A. Makse, Nat. Phys. **6**, 888 (2010).
[17] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. E **83**, 051134 (2011).
[18] V. Rosato, L. Issacharoff, F. Tiriticco, S. Meloni, S. De Porcellinis, and R. Setola, Int. J. Circuit Infrastruct. **4**, 63 (2008).
[19] A. Vespignani, Nature (London) **464**, 984 (2010).
[20] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Nature (London) **464**, 1025 (2010).
[21] Y. Hu, B. Ksherim, R. Cohen, and S. Havlin, Phys. Rev. E **84**, 066116 (2011).

[22] J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, Phys. Rev. E **85**, 066134 (2012).
[23] S.-W. Son, G. Bizhani, C. Christensen, P. Grassberger, and M. Paczuski, Europhys. Lett. **97**, 16006 (2012).
[24] M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, and A. Arenas, Phys. Rev. X **3**, 041022 (2013).
[25] S. Boccaletti, G. Bianconi, R. Criado, C. I. del Genio, J. Gómez-Gardeñes, M. Romance, I. Sendiña-Nadal, Z. Wang, and M. Zanin, Phys. Rep. (2014).
[26] S. Gómez, A. Diaz-Guilera, J. Gómez-Gardeñes, C.-J. Perez-Vicente, Y. Moreno, and A. Arenas, Phys. Rev. Lett. **110**, 028701 (2013).
[27] F. Radicchi and A. Arenas, Nat. Phys. **9**, 717 (2013).
[28] J. Gómez-Gardeñes, I. Reinares, A. Arenas, and L. M. Floría, Sci. Rep. **2**, 620 (2012).
[29] E. Cozzo, A. Arenas, and Y. Moreno, Phys. Rev. E **86**, 036115 (2012).
[30] C. Granell, S. Gómez, and A. Arenas, Phys. Rev. Lett. **111**, 128701 (2013).
[31] K.-M. Lee, J. Y. Kim, W. K. Cho, K.-I. Goh, and I.-M. Kim, New J. Phys. **14**, 033027 (2012).
[32] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett. **109**, 248701 (2012).
[33] K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. Lett. **87**, 278701 (2001).
[34] A. Cardillo, J. Gómez-Gardeñes, M. Zanin, M. Romance, D. Papo, F. del Pozo, and S. Boccaletti, Sci. Rep. **3**, 1344 (2013).
[35] D. Cellai, E. López, J. Zhou, J. P. Gleeson, and G. Bianconi, Phys. Rev. E **88**, 052811 (2013).
[36] A. Halu, S. Mukherjee, and G. Bianconi, Phys. Rev. E **89**, 012806 (2014).