NETWORK SCIENCE

Synchronization in networks with multiple interaction layers

Charo I. del Genio,[1]* Jesús Gómez-Gardeñes,[2,3]* Ivan Bonamassa,[4] Stefano Boccaletti[5,6]

The structure of many real-world systems is best captured by networks consisting of several interaction layers. Understanding how a multilayered structure of connections affects the synchronization properties of dynamical systems evolving on top of it is a highly relevant endeavor in mathematics and physics and has potential applications in several socially relevant topics, such as power grid engineering and neural dynamics. We propose a general framework to assess the stability of the synchronized state in networks with multiple interaction layers, deriving a necessary condition that generalizes the master stability function approach. We validate our method by applying it to a network of Rössler oscillators with a double layer of interactions and show that highly rich phenomenology emerges from this. This includes cases where the stability of synchronization can be induced even if both layers would have individually induced unstable synchrony, an effect genuinely arising from the true multilayer structure of the interactions among the units in the network.

INTRODUCTION

Network theory (*1–9*) has proved a fertile ground for the modeling of a multitude of complex systems. One of the main appeals of this approach lies in its power to identify universal properties in the structure of connections among the elementary units of a system (*10–12*). In turn, this enables researchers to make quantitative predictions about the collective organization of a system at different length scales, ranging from the microscopic to the global scale (*13–19*).

Because networks often support dynamical processes, the interplay between the structure and the unfolding of collective phenomena has been the subject of numerous studies (*20–22*). Many relevant processes and their associated emergent phenomena, such as social dynamics (*23*), epidemic spreading (*24*), synchronization (*25*), and controllability (*26*), have been proven to significantly depend on the complexity of the underlying interaction backbone. Synchronization of systems of dynamical units is a particularly noteworthy topic because synchronized states are at the core of the development of many coordinated tasks in natural and engineered systems (*27–29*). Thus, in the past two decades, considerable attention has been paid to shedding light on the role that network structure plays in the onset and stability of synchronized states (*30–42*).

However, in the past years, the limitations of the simple network paradigm have become increasingly evident, as the unprecedented availability of large data sets with ever-higher resolution levels has revealed that real-world systems can seldom be described by an isolated network. Several works have proved that mutual interactions between different complex systems cause the emergence of networks composed of multiple layers (*43–46*). This way, nodes can be coupled according to different kinds of ties so that each of these interaction types defines an interaction layer. Examples of multilayer systems include social networks, in which individual people are linked and affiliated by different types of relations (*47*), mobility networks, in which individual nodes may be served by different means of transport (*48*, *49*), and neural networks, in which the constituent neurons interact over chemical and ionic channels (*50*). Multilayer networks have thus become the natural framework to investigate new collective properties arising from the interconnection of different systems (*51*, *52*). The multilayer studies of processes, such as percolation (*53–57*), epidemics spreading (*58–61*), controllability (*62*), evolutionary games (*63–66*), and diffusion (*67*), have all evidenced a very different phenomenology from the one found on monolayer structures. For example, whereas isolated scale-free networks are robust against random failures of nodes or edges (*68*), interdependent ones are very fragile (*69*). Nonetheless, the interplay between multilayer structure and dynamics remains, under several aspects, still unexplored, and in particular, the study of synchronization is still in its infancy (*70–73*).

Here, we present a general theory that fills this gap and generalizes the celebrated master stability function (MSF) approach in complex networks (*30*) to the realm of multilayer complex systems. Our aim is to provide a full mathematical framework that allows one to evaluate the stability of a globally synchronized state for nonlinear dynamical systems evolving in networks with multiple layers of interactions. To do this, we perform a linear stability analysis of the fully synchronized state of the interacting systems and exploit the spectral properties of the graph Laplacians of each layer. The final result is a system of coupled linear ordinary differential equations for the evolution of the displacements of the network from its synchronized state. Our setting does not require (nor assume) special conditions concerning the structure of each single layer, except that the network is undirected and that the local and interaction dynamics are described by continuous and differentiable functions. Because of this, the evolutionary differential equations are nonvariational. We validate our predictions in a network of chaotic Rössler oscillators with two layers of interactions featuring different topologies. We show that, even in this simple case, there is the possibility of inducing the overall stability of the complete synchronization manifold in regions of the phase diagram where each layer, taken individually, is unstable.

RESULTS

The model

From the structural point of view, we consider a network composed of N nodes, which interact through M different layers of connections, each layer generally having different links and representing a different kind of

[1]School of Life Sciences, University of Warwick, Coventry CV4 7AL, U.K. [2]Departamento de Física de la Materia Condensada, University of Zaragoza, 50009 Zaragoza, Spain. [3]Institute for Biocomputation and Physics of Complex Systems, University of Zaragoza, 50018 Zaragoza, Spain. [4]Department of Physics, Bar-Ilan University, 52900 Ramat Gan, Israel. [5]CNR–Istituto dei Sistemi Complessi, Via Madonna del Piano, 10, 50019 Sesto Fiorentino, Italy. [6]Embassy of Italy in Israel, 25 Hamered Street, 68125 Tel Aviv, Israel.
*Corresponding author. Email: c.i.del-genio@warwick.ac.uk (C.I.d.G.); gardenes@unizar.es (J.G.-G.)

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