Optimizing diffusion in multiplexes by maximizing layer dissimilarity

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Diffusion in a multiplex depends on the specific link distribution between the nodes in each layer, but also on the set of the intralayer and interlayer diffusion coefficients. In this work we investigate, in a quantitative way, the efficiency of multiplex diffusion as a function of the topological similarity among multiplex layers. This similarity is measured by the distance between layers, taken among the pairs of layers. Results are presented for a simple two-layer multiplex, where one of the layers is held fixed, while the other one can be rewired in a controlled way in order to increase or decrease the interlayer distance. The results indicate that, for fixed values of all intra- and interlayer diffusion coefficients, a large interlayer distance generally enhances the global multiplex diffusion, providing a topological mechanism to control the global diffusive process. For some sets of networks, we develop an algorithm to identify the most sensitive nodes in the rewirable layer, so that changes in a small set of connections produce a drastic enhancement of the global diffusion of the whole multiplex system.

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I. INTRODUCTION

Multilayer networks have been the focus of intense research in recent times [1,2]. Such interest arises mainly from the necessity of exploring new emergent properties in networks whose backbone is formed by different types of connections [3–5]. Many aspects of network theory have been recently revisited under the paradigm of multilayer systems. These features cover the stability of technological interdependent networks that has caused a catastrophic breakdown of energy distribution [6], mathematical aspects related to their spectral properties [7,8], and critical phenomena [9].

During recent years, most of the attention has been devoted to a particular class of multilayer networks called multiplexes. A multiplex is a multilayer structure formed by M layers, each of them being itself a network [5,10,11] and containing exactly the same quantity, N, of nodes. This way, we can represent systems in which a set of N nodes can be connected through links of M types. Such a feature can be effectively observed in a series of actual complex systems, e.g., a set of airports that are connected by different airlines, each one with its own set of flights connecting a common pool of destinations [12,13]. The same is observed for a group of individuals that may communicate with each other through different media [14] or use different communication means with the same set of locations [15–17].

To assemble a multiplex (see Fig. 1), we represent each of the N entities (airports, individuals, locations, etc.) in each of the M layers, so that for each entity there are M nodes (one per layer) that represent it. Each of the N nodes in a layer is directly connected to its $M-1$ counterparts in the other layers as they represent the same entity while the rest of the connections are established within the layer to which they belong.

The particular form of multiplexes and its ubiquity as the backbone of real complex systems has motivated the development of a mathematical framework for their treatment [11]. This has helped the analysis of the emergence of collective behavior such as percolation [18–20], epidemics [21–26], coordination [27,28], cooperation [29–32], and synchronization [33–36], among others [37]. In many cases, these studies have shed light on the new physical phenomena that the coupling between the interaction layers of the multiplexes induces to the collective behavior of such systems [38].

A general issue related to multiplex systems is the understanding of diffusive processes on such a structure, and, particularly, its relation to the diffusive properties of each interaction when considered independently [39]. Multiplex diffusion depends on diffusion within each individual layer, $\alpha = 1,2,\ldots,M$, but also on the interlayer diffusion coefficients $D_{\alpha,\beta}$. If these are all set to zero, multiplex diffusion is restricted to each specific layer, depending only on the specific link distribution in that layer and on the intralayer diffusion coefficients. In the other extreme, the full potential of the multiplex is reached when all $D_{\alpha,\beta} \neq 0$, which allows a direct connection between any pair of layers. Therefore, it is quite a difficult task to predict, in a general way, how a global multiplex diffusive process depends on each of the individual intralayer counterparts.

In this work, we propose to relate the efficiency of the global multiplex diffusion to a quantitative measure for the difference between the topological structure of any pair of layers in the multiplex. In a single layer α, the diffusion efficiency, which depends on the intralayer diffusion coefficient D_α and on the network topology, is usually expressed in terms of the smallest nonzero eigenvalue λ_2^α of the corresponding Laplacian matrix L_α [40]. Similarly, multiplex diffusion is expressed by Λ_2, the smallest nonzero eigenvalue of the *supra-Laplacian* matrix, as will be detailed in the next section. Thus, we investigate how Λ_2 depends on the dissimilarity between layers, measured through the network distance introduced in [41]. In particular, for a multiplex, we can evaluate $M(M-1)/2$ values $\delta(\alpha,\beta)$ corresponding to the distance between a pair of layers α and β. For the sake of a clearer presentation of our results, we restrict our analysis to the simplest situation of an $M = 2$ multiplex to follow in a close way the relation between Λ_2 and the layer distance δ while proceeding with a controlled rewiring of one of the networks.

The rest of this work is organized as follows: In Sec. II, we show how to describe a diffusion process in a multiplex by

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