Controlling Chaotic Solitons in Frenkel-Kontorova Chains by Disordered Driving Forces

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We discuss a general mechanism explaining the taming effect of phase disorder in external forces on chaotic solitons in damped, driven, Frenkel-Kontorova chains. We deduce analytically an effective random equation of motion governing the dynamics of the soliton center of mass for which we obtain numerically the regions in the control parameter space where chaotic solitons are suppressed. We find that such predictions are in excellent agreement with results of computer simulations of the original Frenkel-Kontorova chains. We show theoretically how such a fundamental mechanism explains recent numerical results concerning extended chaos in arrays of coupled pendula [S. F. Brandt *et al.*, Phys. Rev. Lett. **96**, 034104 (2006)].

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Synchronization phenomena and dynamic control of networks of coupled nonlinear oscillators have attracted a great deal of attention in recent years [1] owing to their relevance in diverse fields such as physics [2], biology [3], and chemistry [4], to cite just a few. A fundamental problem closely related to the synchronization-desynchronization transition in the context of chaotic arrays is the control of chaos [5]. Studies have shown that chaos in coupled arrays of periodically forced, damped, nonlinear oscillators can be tamed by parametric disorder [6–9], impurities [10–12], localized controlling resonant forces [13,14], random shortcuts [15], and global disordered driving forces [16]. With respect to the effect of *disorder* on spatiotemporal chaos, it has been proposed [6] that a possible mechanism (hereafter referred to as mechanism I) by which disorder may regularize a chaotic array involves the removal of some of the oscillators from their chaotic band, thus giving rise to subpopulations of regularized oscillators that can readily frequency lock to the external drive. These periodic subpopulations are generally able to impose their rhythm, via the coupling, on the remaining chaotic subpopulations thus driving the whole array to a periodic state. However, it has been shown numerically that disorder can also tame chaos while keeping *all* the oscillators in their chaotic regimes, such as in the case of arrays of chaotic coupled pendula with disordered initial phases in the driving forces [16]. In this Letter, we discuss a basic physical mechanism which explains this intriguing taming effect of disorder. We demonstrate below that the suppression of chaos implies the incidental regularization of *pairs* of adjacent pendula. For the sake of clarity, the general approach is discussed through the analysis of arrays of damped, driven Frenkel-Kontorova (FK) chains $ml^2\ddot{\theta}_n + mgl\sin\theta_n = -\gamma\dot{\theta}_n + \tau' + \tau\sin(\omega t + \varphi_n) + \kappa(\theta_{n+1} + \theta_{n-1} - 2\theta_n)$ which, for convenience, are studied in dimensionless form

$$\frac{d^2u_n}{dt'^2} = -\frac{K}{2\pi}\sin(2\pi u_n) - \alpha\frac{du_n}{dt'} + \Gamma_0 + \Gamma_1\sin(\Omega t' + \varphi_n) + u_{n+1} - 2u_n + u_{n-1}, \qquad (1)$$

where $u_n \equiv \theta_n/(2\pi)$ is the phase of the nth pendulum, $\alpha \equiv \gamma/\sqrt{ml^2\kappa}$ is the damping coefficient, $K \equiv lmg/\kappa$ measures the strength of the substrate potential, $\Gamma_0 \equiv \tau'/(2\pi\kappa)$, $\Gamma_1 \equiv \tau/(2\pi\kappa)$, $t' \equiv t\sqrt{\kappa/(ml^2)}$, $\Omega \equiv \omega\sqrt{ml^2/\kappa}$ are the amplitudes of constant and periodic excitations, and dimensionless time and frequency, respectively. As is well known, the FK model provides a fairly accurate description of diverse physical and biological systems and phenomena, including ladder networks of discrete Josephson junctions, DNA dynamics, and charge density wave conductors, among many others [17]. Here we assume a finite chain of N particles with the following periodic boundary condition: $u_0 = u_N - N - 1$, $u_{N+1} = u_1 + N + 1$ in order to keep the analysis close to experimental possibilities (e.g., a circular array of Josephson junctions). We assume that when driven synchronously (i.e., all periodic excitations have the same initial phase) the chain presents a chaotic soliton at parameter values for which the uncoupled pendula exhibit chaos, and study the disorder-induced regularization effect by randomly choosing the initial phases φ_n uniformly from the interval $[-k\pi, k\pi]$ with $k \in [0, 1]$ being the disorder parameter. As is well known, a collective coordinate formalism (CCF) [18] permits one to describe the motion of the soliton center of mass $X(t)$ by means of an effective ordinary differential equation, which is a perturbed pendulum for the FK model (see [14] and references therein for more details). Thus, the application of CCF to Eq. (1) by assuming a sine-Gordon profile for the discrete soliton, $u_n = n \pm (2/\pi)\tan^{-1}\{\exp[n - X(t)]/l_0\}$, yields the driving force $F_d = -(\Gamma_0/l_0)\sum_n \mathrm{sech}[(n - X)/l_0]/\pi - (\Gamma_1/l_0) \times [A(\varphi_n, k)\sin(\Omega t') + B(\varphi_n, k)\cos(\Omega t')]/\pi$ and hence the family of randomly perturbed pendulum equations (one for each sampling of the uniform distribution)

$$\frac{d^2z}{dt^{*2}} = -\sin z - \delta\frac{dz}{dt^*} - \Gamma_0^* - \Gamma_1^* A(\varphi_n, k)\sin(\tilde{\Omega}t^*) - \Gamma_1^* B(\varphi_n, k)\cos(\tilde{\Omega}t^*), \qquad (2)$$

where $z \equiv 2\pi X$, $t^* \equiv \Omega_{\mathrm{PN}}t'$, $\delta \equiv \alpha\Omega_{\mathrm{PN}}^{-1}$, $\Gamma_0^* \equiv \pi^3 l_0\Gamma_0\Omega_{\mathrm{PN}}^{-2}$, $\Gamma_1^* \equiv \pi^3 l_0\Gamma_1\Omega_{\mathrm{PN}}^{-2}$, and $\tilde{\Omega} \equiv \Omega\Omega_{\mathrm{PN}}^{-1}$, where Ω_{PN} and l_0 are the Peierls-Nabarro frequency and the soliton width, respectively $[\Omega_{\mathrm{PN}}^2 = (\pi^4 + 2\pi^6 l_0^2)/[3l_0\sinh(\pi^2 l_0)]$, $l_0 \approx 1/\sqrt{K}]$. Here, $A(\varphi_n, k) \equiv \sum_n \cos(\varphi_n)\mathrm{sech}[(2\pi n - z)/(2\pi l_0)]$, $B(\varphi_n, k) \equiv \sum_n \sin(\varphi_n)\mathrm{sech}[(2\pi n - z)/(2\pi l_0)]$ are disorder-induced random amplitudes having averages $\langle A(\varphi_n, k)\rangle = \mathrm{sinc}(k\pi)$, $\langle B(\varphi_n, k)\rangle = 0$ and standard deviations $\sigma[A(\varphi_n, k)] = \sqrt{[1 + \mathrm{sinc}(2k\pi)]/2 - \mathrm{sinc}^2(k\pi)}$, $\sigma[B(\varphi_n, k)] = \sqrt{[1 - \mathrm{sinc}(2k\pi)]/2}$, and where $\langle\cdot\rangle \equiv (2k\pi)^{-1}\int_{-k\pi}^{k\pi}(\cdot)d\varphi_n$, $\mathrm{sinc}(k\pi) \equiv \sin(k\pi)/(k\pi)$ [19]. Thus, Eq. (2) permits us to calculate the probability of different disorder-induced behaviors of a soliton which is chaotic in the absence of disorder. An illustrative example is shown in Fig. 1, where the probabilities of a soliton maintaining its chaotic behavior or reaching different periodic attractors after 200 driving periods is shown as a function of the disorder parameter. We found numerically that the statistical properties of the cloud of points accumulated in the $A(\varphi_n, k) - B(\varphi_n, k)$ plane after a large number of different samplings of uniform distributions are accurately described by the aforementioned averages and standard deviations for all $k \in [0, 1]$. Thus, one can consider the *effective* deterministic equation

$$\frac{d^2z}{dt^{*2}} + \sin z = -\delta\frac{dz}{dt^*} - \Gamma_0^* - \Gamma_1^*\,\mathrm{sinc}(k\pi)\sin(\tilde{\Omega}t^*) \qquad (3)$$

to reliably characterize the averaged effect of disorder on the chaotic solitons. When $k = 0$ one recovers the previously studied case of a homogeneous sinusoidal excitation [14], as expected, whereas the effective sinusoidal excitation vanishes in the other limit ($k = 1$). The monotonously decreasing behavior of the function $\mathrm{sinc}(k\pi)$ demonstrates the great suppressory effectiveness of disorder and explains previous numerical results (cf. [16]; see below). Let us assume in the following that the damping and excitation terms in Eq. (3) are small amplitude perturbations of the underlying integrable pendulum; i.e., they satisfy Melnikov's method (MM) requirements [20], and that, in the absence of random initial phases ($\varphi_n = 0$), the perturbed pendulum exhibits homoclinic chaos which corresponds to a chaotic soliton existing in the FK model (1). The application of MM to Eq. (3) gives us the effective Melnikov function (MF) $M_{\mathrm{eff}}^{\pm}(t_0^*) = D^{\mp} \mp A_{\mathrm{eff}}\sin(\tilde{\Omega}t_0^*)$, where $D^{\mp} \equiv \mp 2\pi\Gamma_0^* - 8\delta$, $A_{\mathrm{eff}} \equiv 2\pi\Gamma_1^*\,\mathrm{sinc}(k\pi)\,\mathrm{sech}(\pi\tilde{\Omega}/2)$, and where the minus (plus) sign corresponds to the upper (lower) homoclinic orbit of the unperturbed pendulum. Clearly, a necessary condition for the effective pendulum (3) to exhibit homoclinic chaos is $A_{\mathrm{eff}} > \min\{|D^-|, |D^+|\}$, and hence $A_{\mathrm{eff}} \leq \min\{|D^-|, |D^+|\}$ represents a *sufficient* condition for the

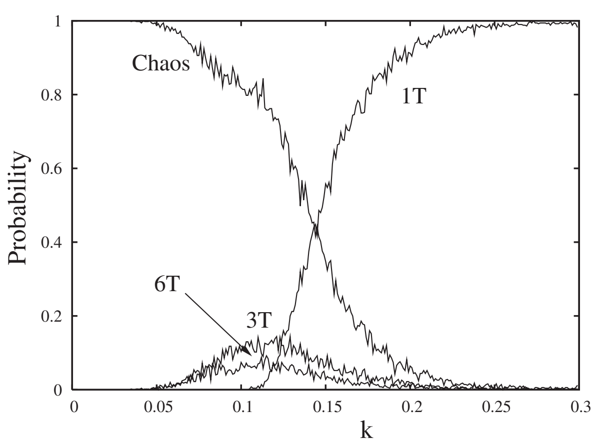

FIG. 1. Probability of chaotic solitons and several regularized behaviors versus the disorder parameter k in a chain of $N = 20$ pendula after averaging over 100 different samplings of the random initial phases. Fixed parameters: $K = 1$, $\alpha = 0.1$, $\Gamma_0 = 0$, $\Gamma_1 = 0.0017$, and $\Omega = 0.04\pi$. Labels indicate periodicity relative to that of the driving.

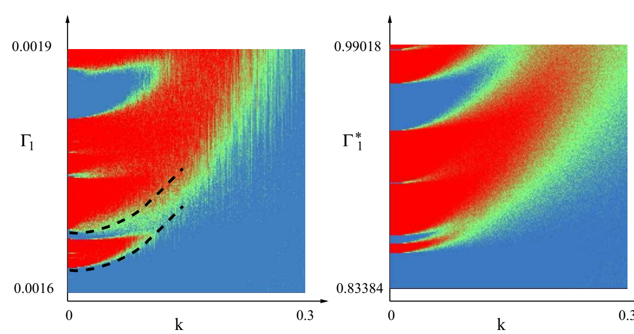

FIG. 2 (color online). Probability of chaotic solitons [dark gray (red): 100%; light gray (green): 50%; medium gray (cyan): 0%] in the $k - \Gamma_1$ and $k - \Gamma_1^*$ parameter planes for the FK chain [Eq. (1), left panel] and for the associated random model from CCF [Eq. (2), right panel], respectively. The function $\Gamma_{1,\mathrm{thresh}} = 0.001\,332/\mathrm{sinc}(k\pi)$ gives the predicted MM-based boundary function for the chaotic threshold [plot not shown, cf. Eq. (3); see the text]. Note that the lower boundaries of the different regularization regions fit the law $\Gamma_1/\mathrm{sinc}(k\pi)$ (dashed black contours), which provides additional support for the effective deterministic model [Eq. (3)]. Only probabilities corresponding to values $k \in [0, 0.3]$ are depicted because of symmetry with respect to $k = 0$. Fixed parameters: $N = 20$ (left panel), $l_0 \approx 1.01\,582$ [to convert Eq. (1) into Eq. (2)] and the remaining parameters are as in the caption of Fig. 1.

suppression of chaos. The equals sign provides the boundary function in the parameter space where homoclinic chaos is suppressed. Next, we compare the analytical predictions from MM and Lyapunov exponent (LE) calculations of both the FK chain [Eq. (1)] and the associated random model [Eq. (2)], with the caveat that one cannot expect too good a quantitative agreement between the two kinds of results because MM is a perturbative method generally related to transient chaos, while LE provides information concerning solely steady chaos. We typically used 100 different samplings of the random initial phases to legitimate the comparison with the predictions from the effective pendulum (3). Typically, one finds for both systems that complete regularization mainly appears inside a region which symmetrically contains the theoretically predicted region where even chaotic transients are suppressed. An illustrative instance is shown in Fig. 2, where the great similarity of both probability distributions for a chaotic soliton in their respective $k - \Gamma_1$, $k - \Gamma_1^*$ parameter planes can be appreciated. Thus, the disordered initial phases tame on average over several different samplings of uniform distributions a chaotic soliton of the chain (1) through the frustration of the homoclinic bifurcation(s) occurring in the associated effective model (3), which is in turn caused by the *disorder-induced decrease* of the effective force's amplitude.

But how does the underlying suppressory mechanism (hereafter referred to as mechanism II) act on a particular sampling of random phases? To explain this, it is convenient to put Eq. (1) into the form

$$\begin{aligned} \frac{d^2 s_n}{dt'^2} + \frac{K}{\pi}\cos(\pi d_n)\sin(\pi s_n) &= -\alpha\frac{ds_n}{dt'} + 2\Gamma_0 + 2\Gamma_1\cos\left[\frac{\varphi_{n+1}-\varphi_n}{2}\right]\sin\left[\Omega t' + \frac{\varphi_{n+1}+\varphi_n}{2}\right] + s_{n+1} - 2s_n + s_{n-1}, \\ \frac{d^2 d_n}{dt'^2} + \frac{K}{\pi}\cos(\pi s_n)\sin(\pi d_n) &= -\alpha\frac{dd_n}{dt'} + 2\Gamma_1\sin\left[\frac{\varphi_n-\varphi_{n+1}}{2}\right]\cos\left[\Omega t' + \frac{\varphi_{n+1}+\varphi_n}{2}\right] + d_{n+1} - 2d_n + d_{n-1}, \end{aligned} \tag{4}$$

where $s_n \equiv u_{n+1} + u_n$, $d_n \equiv u_{n+1} - u_n$. After doubly averaging these equations over the interval $[-k\pi, k\pi]$ $((2k\pi)^{-2} \times \int_{-k\pi}^{k\pi}\int_{-k\pi}^{k\pi}(\cdot)d\varphi_n d\varphi_{n+1})$, one straightforwardly obtains the effective equations

$$\begin{aligned} \frac{d^2 s_n}{dt'^2} + \frac{K}{\pi}\cos(\pi d_n)\sin(\pi s_n) &= -\alpha\frac{ds_n}{dt'} + 2\Gamma_0 + 2\Gamma_1 \mathrm{sinc}(k\pi)\sin[\Omega t'] + s_{n+1} - 2s_n + s_{n-1}, \\ \frac{d^2 d_n}{dt'^2} + \frac{K}{\pi}\cos(\pi s_n)\sin(\pi d_n) &= -\alpha\frac{dd_n}{dt'} + d_{n+1} - 2d_n + d_{n-1}. \end{aligned} \tag{5}$$

In the homogeneous case, when all pendula have the same initial phase ($k = 0$), the effective initial phase difference $(\varphi_{n+1} - \varphi_n)/2 = 0$, $\forall n$, and the initial chaotic state of the chain (whether localized or extended) corresponds to a maximum (minimum) value of the sine (cosine) forcing amplitude in Eq. (4). As disorder increases ($k > 0$), there will be a number of particular pendula, depending upon the specific sampling of random phases, for which the corresponding value of the effective initial phase difference $(\varphi_{n+1} - \varphi_n)/2$ is such that the sine forcing amplitude is now sufficiently small to suppress chaos while the cosine forcing amplitude is not yet sufficiently large to induce chaos. Therefore, mechanism II is no more than the (incidental) regularization of *two* adjacent pendula when their corresponding effective initial phase difference reaches a suitable value for removing the pair of pendula from their chaotic band. Once whichever adjacent pair of the chaotic coupled pendula had been tamed via mechanism II, they could act as a minimal regularized subpopulation from which regularization would extend to the whole chain via mechanism I. Thus, mechanism II explains why the value of k from which regularization of the chain is observed numerically depends upon the particular sampling of random phases [16]: for a fixed k, regularization of adjacent pendula induced by suitable values of their effective initial phase difference can or cannot occur depending upon the *particular* values of the random initial phases $\{\varphi_n\}$ of each sampling. For 2D FK chains, mechanism II predicts that, on average, such a regularization threshold value of k should be lower than the corresponding value for a 1D chain since the number of pendula connected to a given one increases, and hence the chance of a suitable effective initial phase difference also increases on average, as was indeed found numerically in Ref. [16]. Additionally, Eq. (5) allows us to study the synchronization-desynchronization transitions associated with the taming of chaotic states as disorder is increased. Indeed, Eq. (5) indicates that increasing the disorder parameter from $k = 0$ diminishes the force's effective amplitude [again through the function $\mathrm{sinc}(k\pi)$], and hence regularization of the chaotic pendula is generally expected. Thus, a regularization-induced synchronization of the chain, in the sense of optimizing the frequency locking to the periodic excitation, is expected to increase up to a certain critical value k_c. The existence of this critical value is a consequence of two competing effects: a desynchronization effect induced by the constant excitation $2\Gamma_0$ versus a synchronization effect induced by the periodic excitation. Since the former effect remains constant while the latter diminishes as disorder is increased, one expects a monotonous decrease of the degree of synchronization for $k > k_c$. Note that the desynchronization induced by the constant excitation in (5) is a consequence of the multistability existing for sufficiently large values of the disorder parameter: locked and unlocked states coexist for $k > k_c$. We found that this scenario explains previous numerical results [16] and was accurately confirmed by our numerical simulations of Eq. (1) (data not shown).

In conclusion, a basic physical mechanism has been discussed concerning the control by disorder of chaotic solitons as well as extended chaotic states in damped driven FK chains. It should be stressed that such a mechanism is quite general. We have demonstrated theoretically and numerically its effectiveness in a simple realistic situation: disordered initial phases in the driving forces [details of cases not discussed here (e.g., parametric disorder [6–9] and forcing waveform disorder) will be published elsewhere].

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