Stability of metastable structures in dissipative ac dynamics of the Frenkel-Kontorova model

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In this paper we address the question of the survival of metastable structures as steady states of the dissipative dynamics of the Frenkel-Kontorova model. For constant driving force the answer is negative as a consequence of the asymptotic uniqueness of the steady state. On the contrary, when the system is driven by periodic external forces, synchronization to the frequency of the force sustains under some conditions the stable motion of metastable structures. Plausibility arguments leading to this conclusion are confirmed by numerical results on several types of metastable structures. We discuss the applicability of these results to models for charge-density-wave dynamics and Josephson-junction arrays.

I. INTRODUCTION

The set of equilibrium states of many-body models with competing length scales[1] includes not only minimum-energy configurations or ground states but also an overwhelming variety of metastable states. Some of them are simple periodic or quasiperiodic structures, but recently mathematical proofs are available[2] on the existence of truly chaotic metastable states in Frenkel-Kontorova type of models. The stability of these metastable states against (small enough) local fluctuations is assured by the existence of high pinning energy barriers which prevent their relaxation toward stable states of lesser energy.

The dissipative (inertialess) dynamics of this type of many-body model under external forces has received some attention[3] due to its close connection with some physical systems of current interest in condensed-matter physics (charge- and spin-density waves,[4] Josephson-junction arrrays,[5] flux lines motion in layered type-II superconductors,[6] to mention some of them).

The question we address here is whether metastable structures do survive as moving configurations in the dissipative dynamics of this type of model under external forces. In Sec. III we argue that, under suitable conditions, in the presence of time periodic external forces, metastable structures are true steady states of the dissipative dynamics and present numerical evidences of this assertion.

First, in the present introductory section, we introduce the model and the three types of metastable structures on which the analysis will be carried out. Later, in Sec. II, general results on the dissipative dynamics of regular structures (commensurate, incommensurate, and elementary discommensurations) are discussed, before entering into the arguments of Sec. III. In Sec. IV we summarize the results and discuss their applicability to models for charge-density-wave dynamics and Josephson-junction arrays.

Let us consider an array of particles with positions u_j, each one coupled to its nearest neighbors by a convex interaction (which for definiteness we will take as harmonic), and in a periodic pinning potential V

$$V(u) = \frac{K}{(2\pi)^2}\left[1-\cos(2\pi u)\right]. \tag{1}$$

The properties of the ground states depend crucially on the average interparticle distance (winding number), $\omega = \langle u_{i+1} - u_i \rangle$. Incommensurate ground states, which correspond to (sufficiently) irrational values of ω, are sliding at low values of the pinning potential but above certain critical value they become pinned. Commensurate (C) ground states correspond to rational values of $\omega = p_0/q_0$ (p_0, q_0 coprime integers); they are periodic, $u_{n+q_0} - p_0 = u_n$, and pinned configurations; because they are pinned, they admit defects or discommensurations (DC). One can visualize an elementary DC as a localized compression or expansion of a C ground state, with an "excess length" (or phase shift) equal to either $1/q_0$ (advanced DC) or $-1/q_0$ (delayed DC).[7]

If a configuration consists of a C structure of commensurability $\omega_0 = p_0/q_0$, upon which an array of elementary DC's of the same type (either advanced or delayed) is superimposed; its winding number is given by

$$\omega = \omega_0 + c\Delta, \tag{2}$$

where c is the inverse of the average number of particles between contiguous DC's, and Δ is the excess length associated with each DC. If the spacing between DC's is regular, this is a correct phenomenological description of a ground state of commensurability ω (provided $c << 1$), but when the DC's are not regularly separated, the structure is only metastable; hereafter we will refer to this type of structure as type-I metastable structure.

Suppose now that the array of DC's contains an equal number of advanced and delayed DC's. Then, the winding number of the structure is ω_0 (as for the underlying C structure). This is a metastable structure which we will refer to as type II.

The metastable structures of type III are formed by connecting blocks of C structures of different commensurabilities. If ω_1 and ω_2 are the winding numbers of the C structures involved and λ is the proportion (measured in length units) of the first one, the winding number ω of the total structure satisfy

$$\omega^{-1} = \lambda\omega_1^{-1} + (1-\lambda)\omega_2^{-1}. \quad (3)$$

The region where the blocks are connected is usually called an interface.[7]

Figure 1 shows schematically some examples of DC's and metastable states of different types. In Fig. 2 the same are shown using a convenient variable, namely the "relative local phase." If $\{u_i^0\}$ denotes the sequence of particle positions in a C structure of reference ($\omega_0 = p_0/q_0$), and $\{u_i\}$ is the sequence for the DC or metastable state under consideration, the relative local phase sequence φ_j is defined as

$$\varphi_j = \frac{1}{q_0}\sum_{i=0}^{q_0-1}(u_{j+i} - u_{j+i}^0). \quad (4)$$

For the above description of the three types of metastable structures to be a sensible one, it is, of course, necessary that the width of the DC's or interfaces involved be small compared to the spacings between them.

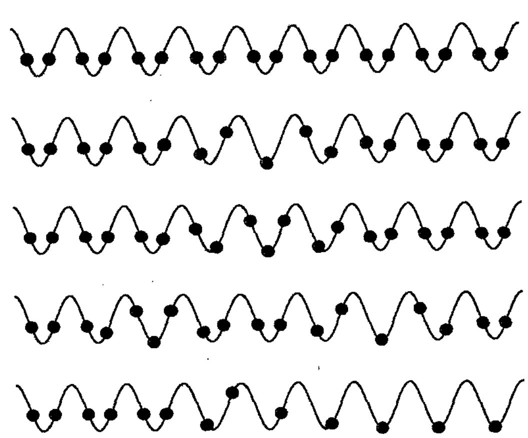

FIG. 1. Schematic representation of different types of states. (a) Commensurate state, $\omega_0 = 1/2$. (b) Advanced DC on a $\omega_0 = 1/2$ state. (c) Delayed DC on a $\omega_0 = 1/2$ state. (d) A type-II metastable structure which consists of an advanced and a delayed DC on a $\omega_0 = 1/2$ state. (e) Type-III metastable structure formed by two commensurate blocks ($\omega_1 = 1/2$, $\omega_2 = 1$).

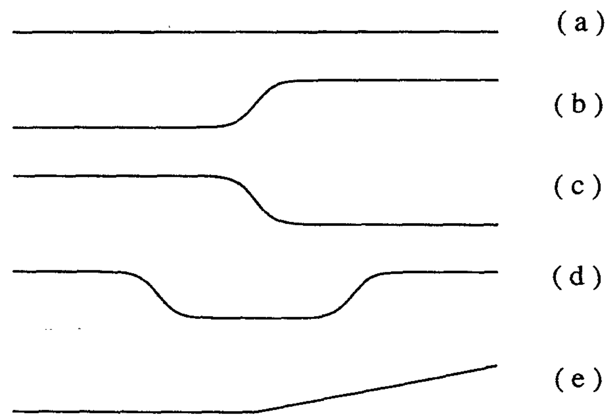

FIG. 2. Schematic representation of the "relative local phase" variable for the five structures shown in Fig. 1. In all the cases the C structure of reference is $\omega_0 = 1/2$.

II. GENERAL FEATURES OF DISSIPATIVE DYNAMICS

The equations of dissipative motion of the system when an external force $F(t)$ is applied can be written as

$$\dot{u}_n = u_{n+1} + u_{n-1} - 2u_n - \frac{K}{2\pi}\sin(2\pi u_n) + F(t). \quad (5)$$

For this type of equation, and with some sensible assumptions on $F(t)$, Middleton[8] has proved that (for fixed model parameters, as well as ω) the (particle and time) average velocity $\bar{v}$ is unique. He was also able to prove the uniqueness of the steady state itself *in the case that the applied force is constant.* Simple additional considerations show that, in this case, the unique steady state is the one which corresponds to moving regular[9] (not metastable) structures. Consequently, in the dissipative dynamics under constant external force, initial metastable configurations always relax toward regular steady states. There is no place for moving complex spatial structures in the dissipative dynamics under constant external force.

For later purposes it is convenient to introduce the notion of velocity $\tilde{v}$ of a DC relative to the underlying C structure. Consider a configuration which consists of an elementary DC upon a C structure, and assume that at time $t = t_0$ the DC is centered at particle j_0. One can naturally choose the center of the DC as where the deviation from the reference C configuration is largest; if this choice is ambiguous, j_0 can be always fixed by some convention. If at some later time $t = t_0 + \tau$, the configuration is related to the initial one by relabeling and integer spatial translations, and the center of the DC is now at particle $j_0 + N$, we will say that the (mean) relative velocity $\tilde{v}$ of the DC is

$$\tilde{v} = \frac{N}{\tau}. \quad (6)$$

In terms of the relative local phase defined by Eq. (4) the DC can be easily identified in the structure and such relative velocity implies that $\varphi_j(t+\tau) = \varphi_{j-N}(t)$. As we see, this definition [Eq. (6)] requires the periodicity of the motion, though extensions to the more general case of recurrent (e.g., quasiperiodic) motion can be consistently made. We note also that the relative velocity is defined as the advance, measured in number of particles, per unit time of the DC; then, the total displacement of the structure that results from the motion of an elementary DC of relative velocity $\tilde{v}$ and of excess length Δ during a time interval τ is $-\tilde{v}\tau\Delta$. Here the sign of the displacement reflects that the motion of an advanced (delayed) DC with $\tilde{v} > 0$ implies a backward (forward) motion of the particles. Then, a configuration consisting of a C structure moving at average velocity $\bar{v}_0$ and a density c of DC's with relative velocity $\tilde{v}$ and excess length Δ, has an average velocity

$$\bar{v} = \bar{v}_0 - \tilde{v}c\Delta. \tag{7}$$

Although for an elementary discommensuration the asymptotic relative velocity $\tilde{v}$ is a well defined (unique) function of the external force, one should not expect it to be independent on the DC's concentration c, when considering the motion of an array of elementary DC's. However, on physical basis, the expectation is that at very low densities ($c << 1$) the DC relative velocity will not differ substantially from the value $\tilde{v}$ corresponding to zero density.

Now, let us consider external periodic forces,

$$F(t) = \bar{F} + F_{\text{ac}}\cos(2\pi\nu_0 t), \tag{8}$$

acting on the system. Numerical studies of the dissipative dynamics under forces [Eq. (8)] have shown[10] that the average velocity $\bar{v}$ of C structures locks at resonant values $\bar{v} = \nu_0(r\omega+m)/s$ (r, m, s integers), i.e., for each resonant value of $\bar{v}$ there is a finite interval of values of $\bar{F}$ for which the function $\bar{v}(\bar{F})$ remains constant. For values of $\bar{F}$ inside the locking intervals, the motion is periodic and the largest Lyapunov exponent of the system trajectories is negative: small fluctuations around the steady-state motion are exponentially damped out; in other words, inside the resonant steps, the particles are synchronized with the external periodic force and this synchronization is robust against small fluctuations.

For incommensurate structures, there are two distinct dynamical regimes. At low values of the pinning potential, the function $\bar{v}(\bar{F})$ does not show resonant steps. For each resonant value of $\bar{v}$ there is a critical value of the pinning parameter K, above which $\bar{v}(\bar{F})$ locks at that resonant value. This transition has been characterized as a dynamical Aubry transition. Below the transition, the steady state admits an analytical description in terms of a smooth hull function, while in the mode-locking regime such analytical description is not possible anymore.[10]

The dissipative dynamics of discommensurations under periodic forces is rather similar to that for incommensurate structures. When we consider values of $\bar{F}$ for which the mean velocity of the underlying commensurate structure remains constant and then study the relative velocity of the DC we find that at low values of the parameter K, the relative velocity as a function of $\bar{F}$, $\tilde{v}(\bar{F})$, is invertible [Fig. 3(a)], but above some critical value of K, $\tilde{v}$ locks at values which are resonant with the frequency ν_0 of the external force [see Fig. 3(b)]. In this mode-locking regime, for values of $\bar{F}$ well inside the locking intervals, both the average velocity of the underlying C structure, $\bar{v}_0$, and the DC relative velocity, $\tilde{v}$, are synchronized; this synchronization is structurally stable, i.e., robust against parameter fluctuations, which is of the utmost importance for the discussion below.

In Fig. 4 we represent the function $\bar{v}(\bar{F})$ for two different commensurate structures defined by $\omega = 1/2$ and $\omega = 6/11$. The latter can be seen as an array of advanced DC's with density $c = 1/11$ upon a simpler $\omega_0 = 1/2$ structure. Even for this (not too low) value of the DC concentration, Eq. (7) fits exactly in all the observable steps.

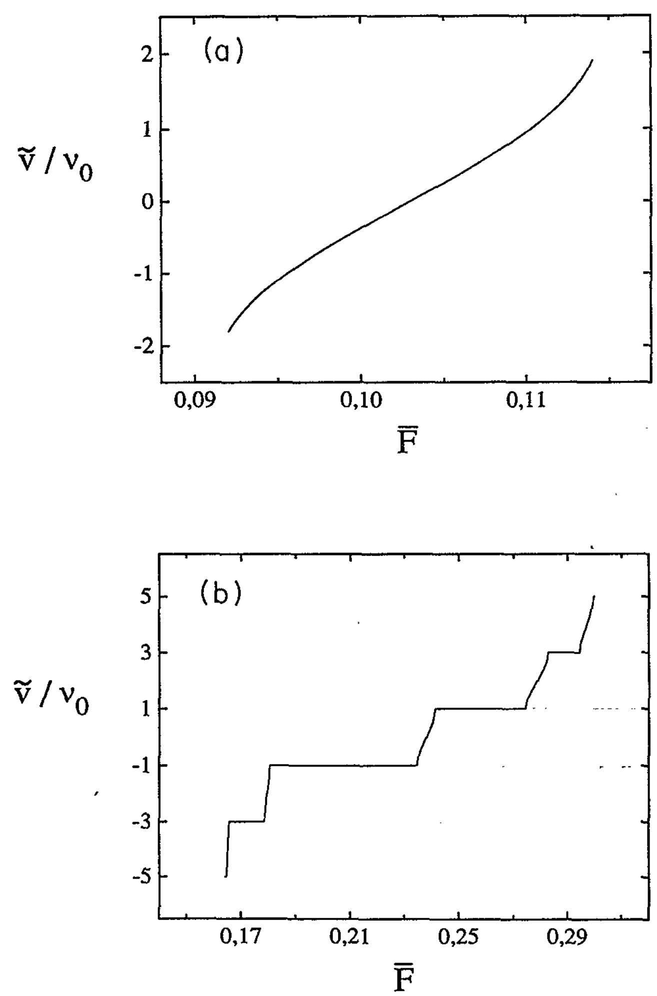

FIG. 3. $\tilde{v}(\bar{F})/\nu_0$ in the case of a delayed DC for two different values of K. (a) $K = 1$ (low K case); $\tilde{v}$ is not locked. (b) $K = 4$ (high K case); $\tilde{v}$ is locked. The underlying structure is a $\omega_0 = 1/2$ structure moving in a locked state. In (b) even steps ($\tilde{v}/\nu_0 = \ldots -2, 0, 2, \ldots$) correspond to subharmonics of the motion of the DC; they really exist but cannot be well appreciated in the figure. ($F_{\text{ac}} = 0.2, \nu_0 = 0.2$.)

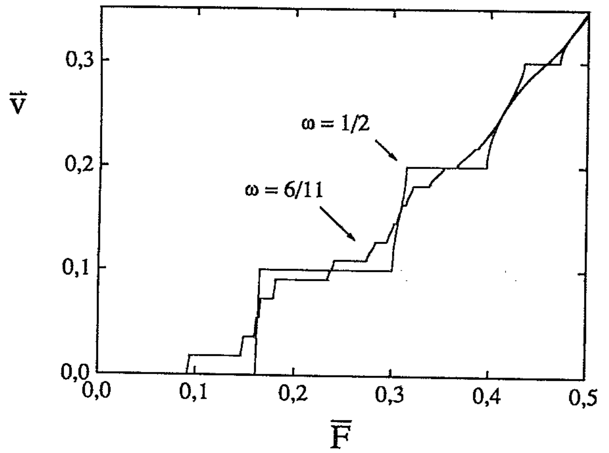

FIG. 4. Characteristic curve, $\bar{v}(\bar{F})$, for two different structures defined by $\omega = 1/2$ and $\omega = 6/11$. ($K = 4.0, F_{ac} = 0.2, \nu_0 = 0.2$.)

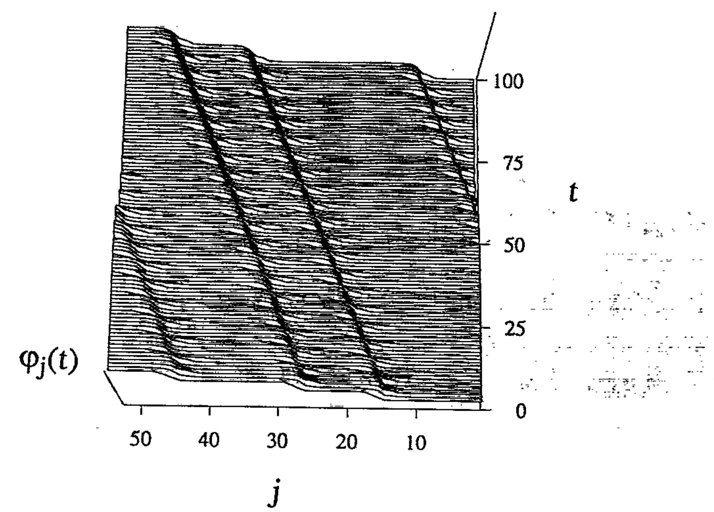

FIG. 5. Moving type-I metastable structure which survives as a true steady state. It consists of a 3/53 concentration of advanced DC's on a $\omega_0 = 1/2$ structure. In this case $\bar{v}_0/\nu_0 = 1/2$ and $\bar{v}/\nu_0 = 25/53$ so $\tilde{v}/\nu_0 = 1$. ($K = 4.0, \bar{F} = 0.2, F_{ac} = 0.2, \nu_0 = 0.2$.)

III. MOTION OF METASTABLE STRUCTURES

Let us now consider a metastable structure of type I and fix $\bar{F}$ (as well as the model parameters) to a value well inside one of the locking intervals of the DC velocity $\tilde{v}$. As the spacing between DC's is irregular, the particle configuration in the DC's is slightly different from the one in the DC's of the regularly spaced structure; however, as far as these deviations are small enough and *provided the $\bar{F}$ value is well inside the locking interval*, the motion of each individual DC proceeds with the locked velocity value. This is what should be expected on the basis of the robustness of mode locking. The synchronous motion of the DC's relative to the commensurate substrate assures then that the irregular spacing between the DC's is preserved as time proceeds, and therefore the metastable structure does survive as a moving metastable configuration without relaxing to a regular array of moving DC's.

Figure 5 shows an example of stable motion of a type-I structure using the variable "relative local phase" defined in Eq. (4). When the external force $\bar{F}$ is tuned out of the locking interval for the relative velocity of the DC, the structure relaxes to a regular array of DC's. Such relaxation can be extremely slow, however. There is also a point here, concerning the assertion of being *out of a locking interval.* Even at moderately high values of the parameter K, the measure (in $\bar{F}$ axis) of locking intervals could be the full measure,[10,11] and then the probability of being out of any locking interval is null.

The stable motion of metastable structures of type II requires the condition that the velocities $\tilde{v}_a$ and $\tilde{v}_b$ of the advanced and delayed DC's relative to the underlying C structure have the same value $\tilde{v}_a = \tilde{v}_b$, on a finite interval of $\bar{F}$. With this proviso, the situation is very much like for the type I above, in the sense that the locked motion of the DC's will preserve the initial distribution of (moderately large) separated DC's. Figure 6(a) shows an example of moving metastable structure of type II. In Fig. 6(b) the value of $\bar{F}$ was fixed out of the "common locking" interval, and the pair of DC's annihilates.

Finally, we consider a metastable structure of type III, made of building blocks of commensurability ω_1 and ω_2, so that the commensurability of the total structure is given by Eq. (3), where λ is the proportion (length) of the ω_1 phase. Let $\bar{v}$, $\bar{v}_1$, and $\bar{v}_2$ denote the average velocities associated to the commensurabilities ω, ω_1, and ω_2, respectively [notice that, for fixed model parameters, $\bar{v}$ is a unique function of ω (Ref. 8)]. In order that the stable motion of the metastable structure be possible, the following kinematical condition:

$$\frac{\bar{v}}{\omega} = \lambda\frac{\bar{v}_1}{\omega_1} + (1-\lambda)\frac{\bar{v}_2}{\omega_2} \qquad (9)$$

must hold on a finite interval of values of $\bar{F}$, of common locking for the three velocities. Provided the interface width is much shorter than their separation, and the $\bar{F}$ value is well inside the interval of common locking, the robustness of the mode-locked steady-state motion of each commensurate block will plausibly prevent the spreading of the interfaces; these will keep their separation, while moving at a mean velocity $\bar{v}_{\text{int}}$:

$$\bar{v}_{\text{int}} = \left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right)^{-1}\left(\frac{\bar{v}_1}{\omega_1} - \frac{\bar{v}_2}{\omega_2}\right). \qquad (10)$$

In order to obtain this last result, it is useful to realize that as the structure moves, particles are passing from one phase to the other. The mean number of particles per unit time crossing the interface (density × relative velocity) can be expressed as

$$\omega_1^{-1}(\bar{v}_1 - \bar{v}_{\text{int}}) = \omega_2^{-1}(\bar{v}_2 - \bar{v}_{\text{int}}) \qquad (11)$$

and from this continuity equation one obtains $\bar{v}_{\text{int}}$ [Eq. (10)].

Figures 7 (a)–(c) show examples of stable and unstable motion of metastable structures of type III. In case (b) the metastable structure relaxes to a different asymptotic metastable structure, while in (c) the homogeneous steady state is reached.

IV. CONCLUSION

The dissipative dynamics of Frenkel-Kontorova models with convex interparticle interactions, under constant external force greater than the threshold (depinning) force, has a unique attractor for all initial conditions, for a given set of parameter values.[8] This asymptotically unique (up to trivial time translations) steady state is a regular[9] moving configuration which admits an analytical description.

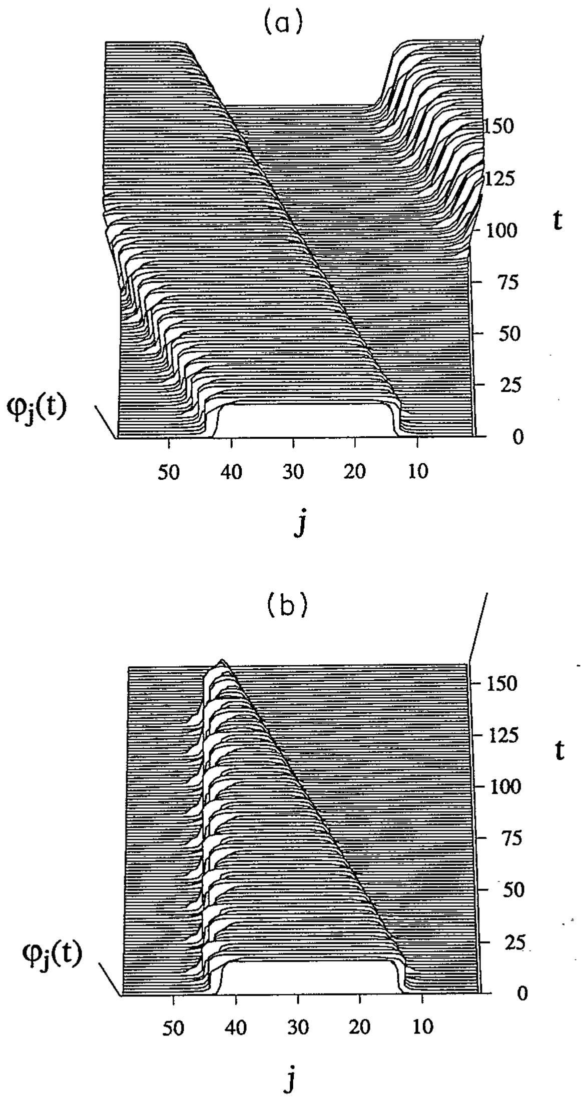

FIG. 6. Type-II structures which consist of two opposed DC's on a $\omega_0 = 2/3$ structure. (a) $\bar{F} = 0.18$; at this value of the external force $\tilde{v}$ is the same for both DC's assuring the stability of the structure. (b) $\bar{F} = 0.164$; each DC moves here with a different value of $\tilde{v}$, extinguishing themselves. ($K = 4.0, F_{ac} = 0.2, \nu_0 = 0.2$.)

On the contrary, when the system is driven by a periodic force, there exist, in general, a rich multiplicity of disequivalent attractors, each having its own basin of initial conditions. All the attractors, however, share a common value for the average velocity. The origin of this phase portrait complexity of the dynamics under time periodic forces, in contrast to the simplicity of the con-

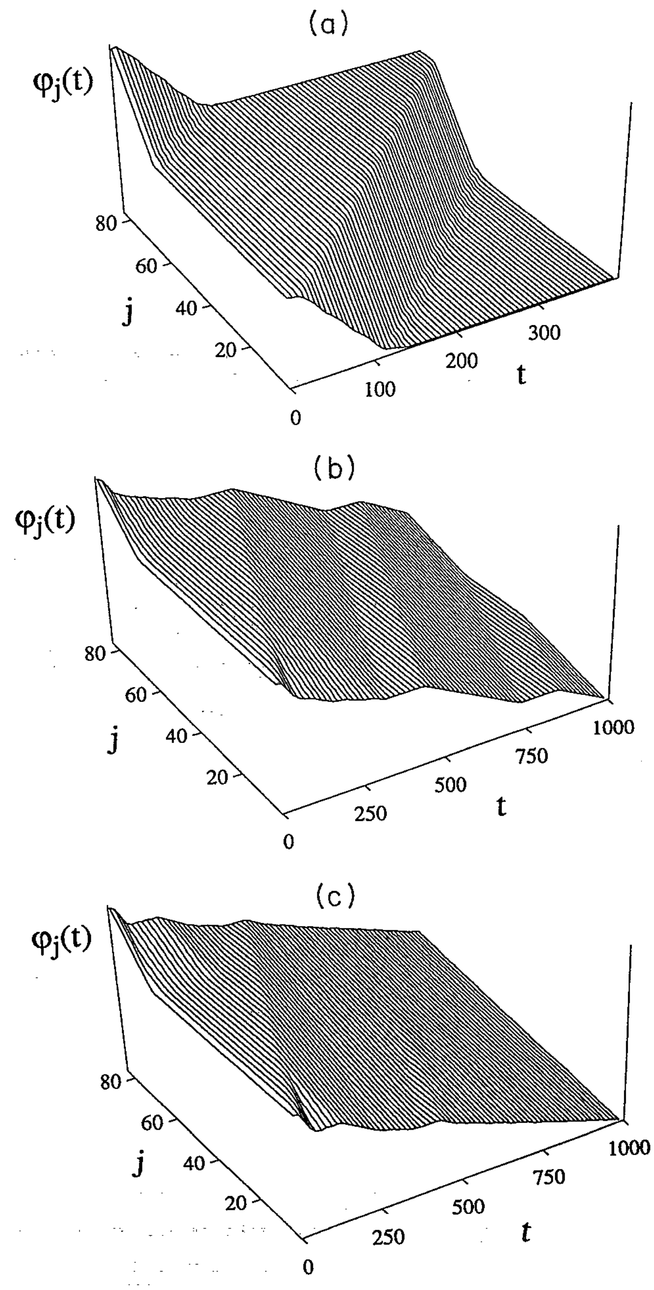

FIG. 7. Three different evolutions for a same initial type-III structure. (a) The initial configuration remains as a true steady state. (b) The initial configuration relaxes to another type-III metastable structure which remains as a true steady state. (c) Here the system relaxes to a regular moving structure. The initial condition is defined by $\omega_1 = 1/2, \omega_2 = 1, \lambda = 32/52$. (a) $\bar{F} = 0.19$, (b) $\bar{F} = 0.22$, (c) $\bar{F} = 0.225$. ($K = 4.0, F_{ac} = 0.2, \nu_0 = 0.2$.) In this picture the commensurate structure of reference chosen to calculate the "relative local phase" variable is a $\omega_0 = 1/2$ structure.

stant force driven dynamics, lies on the synchronization of the motion of defective structures.

In a similar manner as static defective structures of the model are prevented to relax toward minimum-energy configurations by the energy barriers created by the pinning potential, the trajectories corresponding to initial defective structures under time periodic forces are kept off the regular attractor by *dynamical barriers* created by synchronization of the motion to the external force.

Several previous works[12] on the dynamics of discrete elastic chains under periodic driving forces have pointed out the close connection between mode locking and the existence of pinned metastable states. These works focused the analysis on pulsed driving forces of the form

$$F(t) = \begin{cases} F & \text{if } 0 < t \le t_{\rm on} \\ 0 & \text{if } t_{\rm on} < t \le t_{\rm on} + t_{\rm off} = T \\ F(t-T) & \text{if } t > T, \end{cases} \tag{12}$$

where $t_{\rm on}$ is small and $t_{\rm off}$ is very large (these conditions being crucial in their analysis). The point of view in those works is that subharmonic[13] mode locking arises from the presence of many *pinned metastable states.* On the contrary, the connection between metastability and mode locking which emerges from the present work is clearly different: the robustness of mode locking is a necessary condition for the existence of *moving steady-state metastable configurations.* One should notice that regular structures do show subharmonic[13] mode locking under sinusoidal driving forces and then, in this case, one cannot attach the origin of subharmonic mode locking to the presence of many pinned metastable states.

We have considered in this paper three types of metastable structures, simple enough to allow for the analysis. For each of these types we have discussed the conditions for the possibility of their survival as true attractors of the dissipative dynamics under periodic forces, and found excellent agreement with the numerical results. Though numerical studies are restricted to periodic metastable configurations, the plausibility arguments given here apply as well to spatially chaotic arrays of DC's or interfaces; that is to say that chaotic spatial configurations could be observable in the dissipative dynamics of models with convex interparticle interactions.

Dynamical defective structures in the vortex lattice have been shown in recent experiments on Josephson-junction arrays.[14] Numerical simulations of the dynamics of these systems also show the persistence of defective structures as true steady states. However, the dynamical stability of these metastable structures cannot be unambiguously adscribed to the synchronization mechanism described here: first, the interaction energy between phases of neighbor superconducting islands is nonconvex, and second, the equations of motion for the RSJ (resistively shunted junctions) model of the array contains a sort of global coupling between phase velocities. Both features constitute major differences between the Josephson array and the Frenkel-Kontorova model regarding the issue of the multiplicity of attractors in the dynamics of these models.

Our results are directly applicable to the Fukuyama-Lee-Rice (FLR) model of CDW (charge-density-wave) dynamics,[11] but not to modifications of it[15] which incorporate a global coupling between velocities. Consequently, the FLR model under combined dc and ac forces exhibits multiple disequivalent mode-locked states, for a given set of parameters, though all of them display the same average velocity value, and no hysteresis can be observed in the *I-V* curves.

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