Discrete breathers in Josephson arrays

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(Received 23 September 2002; accepted 4 December 2002; published 22 May 2003)

Since the original proposal of 1996 by Floría *et al.* [Europhys. Lett. **36**, 539 (1996)] of intrinsic localization in Josephson ladders, many efforts have been devoted to the theoretical, numerical, and experimental study of such dynamical states in Josephson arrays. Such efforts have already produced around 20 papers on the subject. In this article we will try to review the basic aspects of the physics of discrete breathers in Josephson arrays. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1541131]

One of the concepts from nonlinear science with more important consequences when applied to condensed matter physics is that of *coherent structures*. Linear models of crystals, which have been instrumental in developing a physical understanding of the solid state, cannot explain many other important properties. For example, spatially or temporally coherent structures appear in many nonlinear extended systems. In the last few years, these notions have become fundamental for understanding different problems and their implications extend over many research areas. For instance, it was discovered and proved that in discrete systems nonlinearity may lead to localized vibrations in the lattice that cannot be analyzed using the standard plane wave approach. These intrinsic localized modes have been termed discrete breathers and because they are generic modes in nonlinear lattices, they are the object of great theoretical and numerical attention in diverse fields. Recently, two experimental groups have reported the detection of intrinsic localized modes in superconducting arrays. In this article we will review the basic aspects of the physics of discrete breathers in Josephson arrays.

I. INTRODUCTION

One of the more novel solutions in nonlinear lattices are the intrinsic localized modes or discrete breathers (DB's), a dynamical solution for which energy remains sharply localized in a few sites of the lattice.[1] Arrays made of Josephson junctions (JJ's) are one of the experimental systems where such states have been recently excited and detected.[2,3]

From a nonlinear dynamics perspective a Josephson junction biased by an external current behaves as a forced and damped pendulum. Similarly, a set of superconducting islands coupled by Josephson junctions is modeled as a set of coupled pendula.

In the work by Floría *et al.*[4] a particular JJ configuration, the ladder, biased by ac currents was proposed as an ideal experimental system to study the phenomenon of intrinsic localization. They numerically found two classes of attracting time-periodic solutions whose energy is exponentially localized: oscillator localized modes (or *oscillobreathers*), in which the amplitude of the oscillations of the phases is localized; and rotor localized modes (or *rotobreathers*) where one phase rotates meanwhile the other describe forced oscillations.

Soon after, Sepulchre and Mackay[5] proved the existence of DB's in dissipative networks of weakly coupled oscillators. This work deals mainly with autonomous systems, but in the discussion and conclusions section the authors sketch a proof for the case of ac forced lattices.

These works prove the existence of DB's in dissipative lattices and propose JJ arrays as an adequate experimental system to excite and detect such modes. In what follows we will review the theory and experiments made up to now which show the existence of localized modes in different types of Josephson arrays. Please also see the review articles by Ustinov *et al.* and Fistul *et al.* in this issue.

We will start with the dynamics of a single JJ biased by an external current. Then we will consider Josephson arrays and introduce a framework to the study of the arrays. In Sec. IV we study the case of a JJ parallel array, where only oscillating localized modes can be found. Section V is the main section of the review, there we consider Josephson ladder arrays. In Sec. VI we review the work in single plaquette Josephson arrays. Section VII is devoted to the recent prediction of the existence of DB's in the dynamics of two-dimensional JJ arrays. We finish with a conclusion section.

II. THE SINGLE JJ CIRCUIT

A Josephson tunnel junction is formed by two superconducting electrodes separated by a thin insulating barrier. For the case of junctions in the so-called classical regime, the behavior of the junction is described in terms of the gauge-invariant phase difference variable, φ. The basic equations for the Josephson effect show that the supercurrent through the junction is given by

$$I_s = I_c \sin\varphi, \quad (1)$$

with I_c the critical current or maximum supercurrent of the junction. The voltage through the junction is

$$V(t) = \frac{\Phi_0}{2\pi}\frac{d\varphi}{dt}, \quad (2)$$

where $\Phi_0 = h/2e = 2.07\times10^{-15}$ Wb is called the flux quantum. The phase difference is defined as $\varphi = \theta_1 - \theta_2 - (2\pi/\Phi_0)\int_1^2 \vec{A}\,\vec{d}l$, where $\vec{A}$ is the vector potential and $\theta_{1(2)}$ the phases of the wave function of the superconducting state at each side of the junction.

A single junction in an electrical circuit can be described by the RCSJ (resistively and capacitively shunted junction) model. In this model the junction consists of three branches: one capacitive, one resistive and one superconducting, connected in parallel. The total current through the junction is

$$I = C\frac{dV}{dt} + \frac{V}{R} + I_c \sin\varphi. \quad (3)$$

If we write the equation in terms of the phase difference and normalize current by I_c and time by the plasma frequency $\omega_p = \sqrt{2\pi I_c/\Phi_0 C}$, we get

$$i = \ddot{\varphi} + \Gamma\dot{\varphi} + \sin\varphi = \mathcal{N}(\varphi), \quad (4)$$

where we have defined the nonlinear operator $\mathcal{N}(\varphi)$ and the dissipation or damping parameter of the junction is given by $\Gamma = \sqrt{\Phi_0/2\pi I_c C R^2}$.

Typical values for tunnel junctions involved in the DB experiments are a critical current density of 1 kA/cm^2 and capacitance per unit area of 3 μF/cm^2 which gives a plasma frequency $\omega_p/2\pi$ close to 100 GHz. Thus it is not possible to follow the instantaneous dynamics of the phases. A quantity which is easy to measure is the dc voltage of the junction V (time averaged voltage), which is directly related with the mean velocity of the phase, $V/\omega_p = (\Phi_0/2\pi)\langle\dot{\varphi}\rangle$, also $V/I_c R = \Gamma\langle\dot{\varphi}\rangle$.

Equation (4) is also the equation for the dynamics of the phase of a forced and damped pendulum. Thus, the dynamics of a tunnel junction biased by external currents is similar to the behavior of a force and damped pendulum. This simple analog between the junction and the mechanical pendulum turns out to be very useful to understand the main dynamical properties of the system.

For junctions with an external resistive shunt the effective resistance is the shunted resistance. However, for our unshunted junctions, the resistance is a nonlinear function of the voltage and can be modeled as having a large value at below the gap voltage, rising sharply near the gap voltage, and then remaining the normal state resistance above the gap. This extra structure due to the nonlinear resistance will be reflected in the experimental data (see inset of Fig. 1). In what follows we will sketch the simplest properties of a JJ with a linear resistor and biased by external currents.

A. dc bias

Let us first consider a junction biased by a constant current $i_{dc} = I_{dc}/I_c$, then

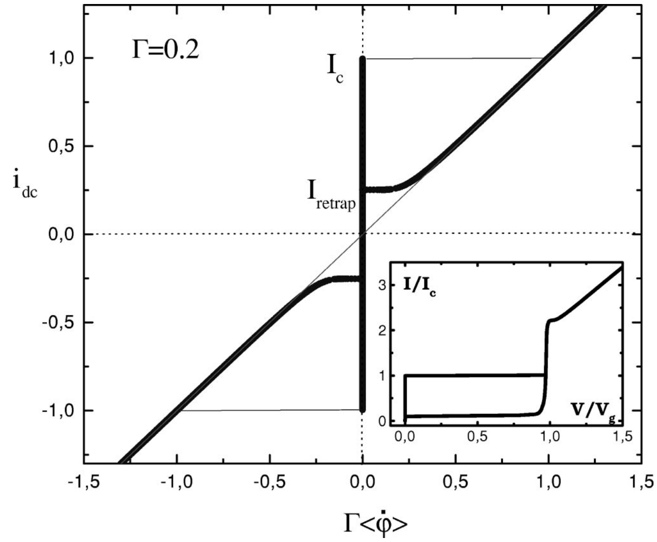

FIG. 1. IV curve for a JJ with a linear resistor biased by a dc current. Two solutions coexist for currents between the critical and the retrapping currents (in this case damping was set to $\Gamma = 0.2$ so $I_r \simeq 0.25 I_c$). The inset shows an experimental IV curve for a tunnel junction. The resistance of the junction below the gap voltage (subgap resistance) clearly differs from the resistance above the gap voltage (normal resistance).

$$\ddot{\varphi} + \Gamma\dot{\varphi} + \sin\varphi = i_{dc}. \quad (5)$$

Two different states are observed. For currents below the junction critical current ($0 < i_{dc} < 1$) a solution of Eq. (5) is the superconducting state of $V = 0$ and $\varphi = \sin^{-1} i_{dc}$. For currents larger than the critical current the phase *rotates* with $V \neq 0$ and the junction is in a resistive state. Depending on the value of the damping and the initial condition this resistive state exists also below the critical current and down to values of the current larger than the so-called junction retrapping current I_r. For sufficiently underdamped junctions I_r can be estimated easily from energy considerations and $I_r/I_c \simeq 4\Gamma/\pi$, also $\langle\dot{\varphi}\rangle \sim i_{dc}/\Gamma$. Thus, for values of the current $I_r < I_{dc} < I_c$, underdamped junctions show an hysteresis loop where two different attractors, one superconducting with $V = 0$ and the other resistive with $V \neq 0$ coexist (see Fig. 1).

B. rf fields

The case of a junction biased by an ac current is more complex and depending on the parameter values and initial conditions different amplitude oscillations, periodic phase rotations and chaotic solutions are found. Now with $I = I_{ac}\sin\omega_{rf}t$ and $\omega = \omega_{rf}/\omega_p$, the normalized governing equation is

$$\ddot{\varphi} + \Gamma\dot{\varphi} + \sin\varphi = i_{ac}\sin\omega\tau. \quad (6)$$

In what follows we will consider three different cases of interest for our purpose.

Figure 2(a) shows a situation where two different amplitude periodic oscillations coexist for certain values of the junction parameters. In both cases $\langle\dot{\varphi}\rangle = 0$ so we have plotted the value of $v_{ac} = \sqrt{2\langle\dot{\varphi}^2\rangle}$ which is larger for the case of

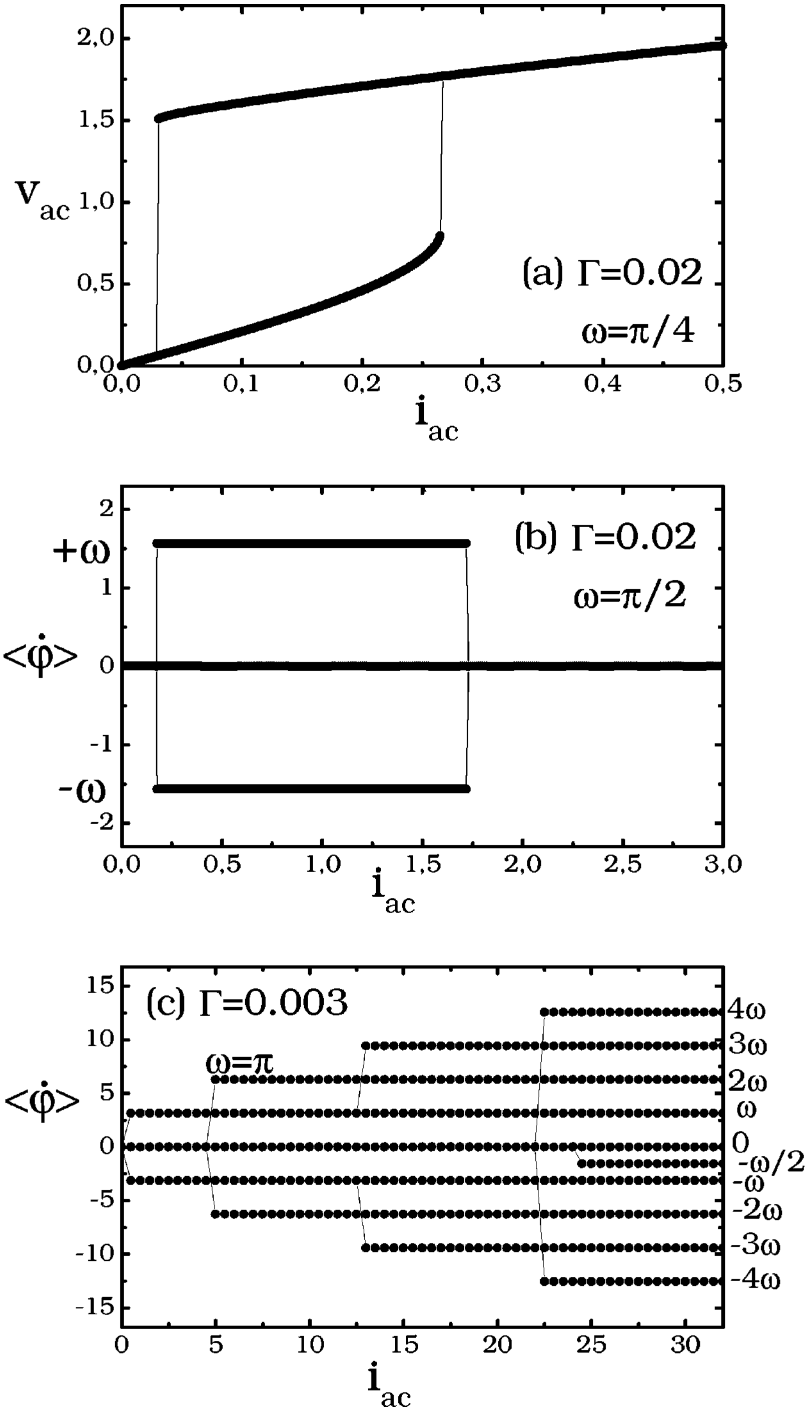

FIG. 2. Underdamped junction biased by an rf field. (a) $\Gamma=0.02$ and $\omega=2\pi\times 0.125$, at a certain value of the amplitude of the field, a "small" amplitude oscillating state destabilizes to a "large" amplitude one. Both states coexist for certain range of values of i_{ac}. (b) $\Gamma=0.02$ and $\omega=2\pi\times 0.25$, for a certain range of values of i_{ac} an oscillating state coexist with two rotating states with $\langle\dot{\varphi}\rangle=\pm\omega$. (c) $\Gamma=0.003$ and $\omega=2\pi\times 0.5$, depending on the value of i_{ac} different attractors coexist with rotation velocities in general given by $\langle\dot{\varphi}\rangle=(n/m)\,\omega$. [The figure shows only one of the possible subharmonic ($m>1$) states.]

large amplitude oscillations. Therefore we observe two different amplitude solutions which coexist for certain range of i_{ac}.

Figure 2(b) shows a situation where an oscillating state ($\langle\dot{\varphi}\rangle=0$) can coexist with two rotating states synchronized to the external field ($\langle\dot{\varphi}\rangle=\pm\omega$).

Figure 2(c) shows a situation where an oscillating state ($\langle\dot{\varphi}\rangle=0$) can coexist with many different rotating states synchronized to the external field ($\langle\dot{\varphi}\rangle=(n/m)\,\omega$).

III. JJ ARRAYS

By coupling JJ with superconducting leads it is possible to make arrays of desired size and geometry. Such arrays behave as coupled nonlinear oscillator lattices. Thus, Josephson arrays are appropriate experimental systems for the study of the phenomenon of intrinsic localization.

To derive general equations for the dynamics of JJ arrays one has to consider Kirchoff's conservation laws (for the current and for the voltage) and the fluxoid quantization condition. Fluxoid quantization demands that for any loop l in the array (with at least one junction) the sum of all the gauge-invariant phase differences around the loop is

$$\sum_{j\in l}\varphi_j=2\pi(n_l-f_l). \tag{7}$$

The integer number n_l results from the multivaluedness of

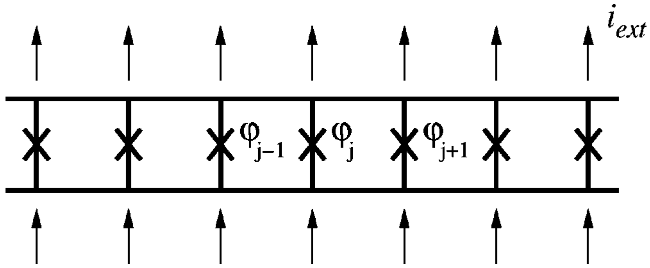

FIG. 3. JJ parallel array biased by a external current. Junctions in the array are represented by crosses.

the superconducting phase variables (θ). However, since in the RCSJ model the dynamical equations depend only on $\ddot{\varphi}$, $\dot{\varphi}$, and $\sin\varphi$ the equations of motion are independent of the n_l and we will eliminate them from our equations in what follows. The term f_l accounts for the total magnetic flux through the loop, measured in units of the magnetic flux quantum Φ_0. It has two contributions: the flux due to external fields applied to the array; and the flux induced by the currents flowing in the system. In general, to compute the total induced flux in a cell of the array one must take into account a full inductance matrix with contributions from all the mesh currents in the array. However, in many cases we can work with the simplest approximation of taking into account only the self-inductance (L) of each cell. In this case for any simple mesh of the array we will write

$$\sum_{j\in m}\varphi_j=-2\pi(f_m^{\text{ext}}+f_m^{\text{ind}})=-2\pi f_m^{\text{ext}}-\frac{1}{\lambda}\frac{I_m}{I_c}, \tag{8}$$

where I_m is the mesh current and $\lambda=\Phi_0/2\pi L I_c$ with L is the self-inductance.

IV. THE JJ PARALLEL ARRAY

One of the simplest Josephson arrays containing many junctions is the uniform JJ parallel array, see Fig. 3. Within the framework of the RCSJ model this array is a physical realization of a one-dimensional lattice of pendula coupled by torsional springs. The dynamical equation for the evolution of the phases are given by

$$\mathcal{N}(\varphi_j)=\lambda(\varphi_{j+1}-2\varphi_j+\varphi_{j-1})+i_{\text{ext}}. \tag{9}$$

For an open array with N junctions (and zero external magnetic field) the boundary conditions are $\varphi_0=\varphi_{N+1}=0$. In the case of circular arrays we impose periodic boundary conditions $\varphi_{j+N}=\varphi_j+2\pi M$ where M is the number of kinks or fluxons trapped in the array.

Equation (9) corresponds to the widely studied Frenkel–Kontorova model or discrete sine-Gordon equation. It is one of the simplest models where localized modes have been proven to exist (see for instance the article by Martínez *et al.* in this issue). In this case the array is biased by an *ac field* and a typical DB mode corresponds to a solution where one of the phases describes a large amplitude oscillation. Meanwhile, the others describe small amplitude oscillations. This state has been called *oscillobreather*. Mathematically, the $\lambda\to 0$ limit of the system corresponds to a set of uncoupled pendula. There, to build the breather, we need two coexisting oscillating attractors, for instance the states shown in Fig. 2(a). In this limit the localized solution is trivial and the existence theorems show us that we can continue such solutions when increasing from zero the value of the coupling.

Then, for some values of the junction parameters and for some range of values of the coupling, DB's should exist in JJ parallel arrays biased by adequate ac fields. However the excitation and detection of such solutions have not been experimentally done up to now. The main problem is related with the lack of any dc signal in the system showing the existence of the DB state. In the case of the oscillobreather all the phases oscillate around equilibria with zero mean voltage. The frequency of such oscillations is very high and then it is not possible to follow the instantaneous voltage in any junction (such measurement would allow unambiguous detection of the DB). A possible operation for detecting the solution would be a measurement of the local power of the radiation generated by the junctions in their oscillating dynamics. However, such experiment has not been tried yet.

A more experimentally accessible possibility is the excitation of *rotobreathers* in JJ arrays. A rotobreather corresponds to a solution where one junction rotates at a given velocity meanwhile the others librate. Then the dc voltage for any librating junction is zero, while for the rotating one will be different from zero. Such a solution is easy to detect by measuring the local dc voltages throughout the array: thus all the efforts to date in detecting DB's in Josephson arrays have been focused in detecting rotobreather states. Another important advantage of the rotobreather states is that in principle they can be obtained by either biasing the array with ac currents [based on the coexistence of attractors shown in Figs. 2(b) and 2(c)] or with a dc current (Fig. 1). This last possibility requires a simpler experimental approach.

One may wonder if these rotating localized modes exist in JJ parallel arrays. In principle, in the uncoupled limit of the model, they can be easily built for the case of dc or ac external bias. However, the convex character of the coupling between junctions in the array shows that any rotating mode will not be stable since the difference between neighboring phases can not grow without limits. Physically, in a parallel array the dc voltage is the same for all the junctions since they are connected by superconducting leads, thus preventing dc voltage localized solutions. To have rotobreathers in Josephson arrays we will need nonconvex interaction terms between neighbors.

V. THE JJ LADDER

A. The system

The simplest way to overcome the difficulty presented in the preceding section is to substitute the horizontal wires connecting neighboring junctions by new JJ elements. Such a new configuration is know as Josephson ladder (see Fig. 4). From our perspective a Josephson ladder can be thought of as a set of parallel pendula, the vertical junctions of the ladder, coupled by sinusoidal terms (the nonconvex terms) provided by the horizontal junctions. The intensity of the coupling is governed by the ratio of the critical current for the horizontal junctions I_{ch} to the critical current for the

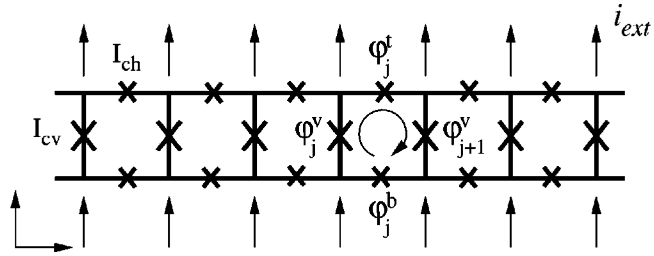

FIG. 4. Anisotropic ladder array with uniform current injection. Vertical junctions have critical current I_{cv} and horizontal junctions I_{ch}.

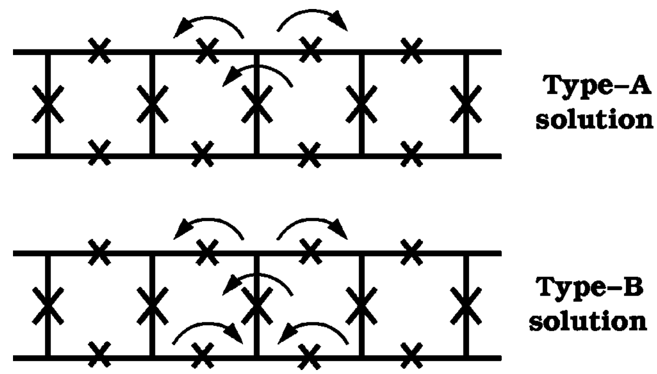

FIG. 5. Two different voltage patterns for a one-site rotobreather in a Josephson ladder. Arrows represent rotating junctions, junctions with dc voltage different from zero.

vertical junctions I_{cv}. Thus, it is natural to study anisotropic ladders where the anisotropy is controlled by the parameter $h = I_{\mathrm{ch}}/I_{\mathrm{cv}}$. A large value of h means that coupling between vertical junctions will be strong meanwhile a small value of h means weakly coupled vertical junctions. Anisotropic arrays are fabricated by varying the area of the junctions. Critical current and capacitance are proportional to this area and the resistance is inversely proportional to it. Thus, also $h = C_h/C_v = R_v/R_h$ and the damping Γ and the plasma frequency ω_p are the same for all the junctions.

The equilibrium properties of infinite ladders were studied in Ref. 6 where it was proved that only the convex part of the interactions are relevant and that the equilibrium properties of the model are qualitatively analogous to those of the parallel array. However, as we will see, the case is different for the dynamics of the system and indeed, rotobreather states exist in Josephson ladders, meanwhile they are prohibited for parallel ones.

Using the RCSJ model and the self-inductance approximation, we can derive the equations of an anisotropic ladder biased by external currents. From the fluxoid quantization condition it will be useful to define the number

$$\xi_j = \varphi_j^v + \varphi_j^t - \varphi_{j+1}^v - \varphi_j^b = -2\pi f_j^{\mathrm{ind}}. \tag{10}$$

In this equation we have considered $f_j^{\mathrm{ext}} = 0$. Nonzero external fields can be easily added to the model [see Eq. (8)]. Rotobreathers in the ladder also exist in the presence of external magnetic fields.

The mesh current is simply $i_j^m = 2\pi\lambda f_j^{\mathrm{ind}} = -\lambda\xi_j$ and the resulting equations can be written compactly as

$$\begin{aligned} \mathcal{N}(\varphi_j^t) &= -\frac{\lambda}{h}\xi_j, \\ \mathcal{N}(\varphi_j^v) &= \lambda(\xi_{j-1} - \xi_j) + i_{\mathrm{ext}}, \\ \mathcal{N}(\varphi_j^b) &= \frac{\lambda}{h}\xi_j, \end{aligned} \tag{11}$$

where open boundaries conditions are imposed by setting $\xi_0 = \xi_{N+1} = 0$. The anisotropy parameter appears after normalizing the current by the vertical junction critical current I_{cv}.

A rotobreather in the ladder corresponds to a localized voltage solution. A set of junctions are in a resistive state with non zero time mean voltage ($V_j \neq 0$) meanwhile the others librate with $V_j = 0$ around an equilibrium state. Such solutions have been numerically found in the ladder for the case of dc and ac bias currents. Although first works dealt with the ac case,[4,7–9] the possibility of exciting and detecting rotobreather solutions in ladders biased by dc external currents it was proposed[10,11] soon afterwards.

B. Theoretical proposal

Before the experiments were done, two different works studied numerically the possibility of detecting rotobreather solutions in dc biased Josephson ladder arrays.

Flach and Spicci[10] studied a ladder with periodic boundary conditions. They restricted the dynamics of the system to the subset of symmetric solutions defined by $\{\varphi_j^t\} = \{-\varphi_j^b\}$ in Eqs. (10) and (11). Their simulations were done in arrays with $\lambda = 1.0$, $h = 1.0$, and $\Gamma = 0.1$. They found rotobreather solutions, proposed a method to excite them and studied the dynamics of the tails of the solutions and some aspects of the resonances between the breather solution and the ladder linear modes.

In Ref. 11 we studied the dynamics of the ladder using Eqs. (11) and found the existence of rotobreather states which can be obtained by continuation from the small h limit. Simulations were done at $\Gamma = 0.2$ and many different values of h and λ. We also proposed a simple method to experimentally excite and detect such solutions in real arrays. Interestingly we found the existence of two types of rotobreather solutions, what we called type-A and type-B solutions. They correspond to two different voltage patterns in the array (see Fig. 5). Type-B solutions comply with an up–down symmetry of the system equations and thus we find one vertical rotating junction with voltage V_{rot} and four horizontal rotating ones with voltages $\pm V_{\mathrm{rot}}/2$. Type-A solutions are asymmetric being characterized by one vertical rotating junction with voltage V_{rot} and two horizontal rotating ones with voltages $\pm V_{\mathrm{rot}}$. We also studied the stability against thermal fluctuations of the solutions and the method to excite them. We studied open arrays of different sizes and saw that rotobreather states can be also excited in single square plaquettes. In this case, assuming a left–right symmetry in the equations and due to the exponential character of the localization phenomenon, we showed that the dynamics of rotobreathers in arbitrary size ladders can be approximately

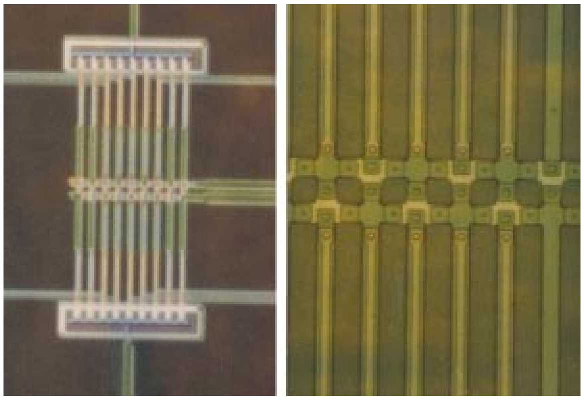

FIG. 6. (Color) Josephson ladder designed to observe the rotobreather states.

mapped to the dynamics of such states in a single square plaquette, once the coupling parameters, h and λ, are conveniently renormalized.

C. The experimental detection

On the basis of these theoretical approaches the experimental effort to excite and detect localized solutions in dc biased Josephson ladders was carried out independently by the group of Ustinov in Erlangen[3] and the group of Orlando at MIT.[2] The experiments were done in anisotropic ladders fabricated from niobium tunnel junctions which have aluminum oxide as the insulating barrier. After these works more experiments were done and the results reported on various publications.[12–17]

Experiments at MIT[2] were done in open ladders with nine vertical junctions (see Fig. 6). The anisotropy parameter of the arrays was $h=1/4$. Ladders were measured at temperatures from 4.2 to 9.2 K, which implies $0.03<\Gamma<0.6$ and $0.04<\lambda<0.45$. We found DB's for $T<6.7$ K which corresponds to $\Gamma<0.2$ and $\lambda<0.05$. The ladder was biased by a dc external current and a set of bias resistors assured a good level of uniformity in the bias. Breathers were excited using an extra current source applied locally at the center of the array. This extra current was eliminated once the rotobreather was excited. To detect the breather we measured local dc voltages at many different points of the array.

Figure 7 shows the result of two experiments where a DB solution was excited at a current close to 1.4 mA in the central junction of a ladder with nine vertical junctions. First the current was increased and the breather disappeared at current values close to the junction critical currents where the other vertical junctions jumped to the resistive state. In the second case the current was decreased. The experiment showed a cascade of transitions from the original one-site rotobreather to other multi-site breather states and finally to the zero voltage state at small enough currents. Such transitions are related with junction retrapping mechanisms.

In addition to the type-B solutions shown, we also found the asymmetric (type-A) states predicted by numerical simulations in Ref. 11.

Finally, by measuring at different temperatures we made an existence phase diagram of localized solutions in the current-damping plane and for $h=0.25$ and $\lambda\simeq0.05$.

In Ref. 3 Binder *et al.* showed measurements in an annular ladder at 4.2 K. The anisotropy of the ladder was $h=0.44$, $\lambda=0.37$ and damping between 0.1 and 0.02. It is important to mention that these arrays have a value of λ almost one order of magnitude larger than those measured at MIT. Rotobreather solutions were also excited and observed and they also found a cascade of switching transitions to multi-site states when decreasing the current. In their experiments the DB's were detected by local dc voltage measurements and by using the method of low temperature scanning laser microscopy which allows for nice visual observations of the different patterns excited in the arrays.

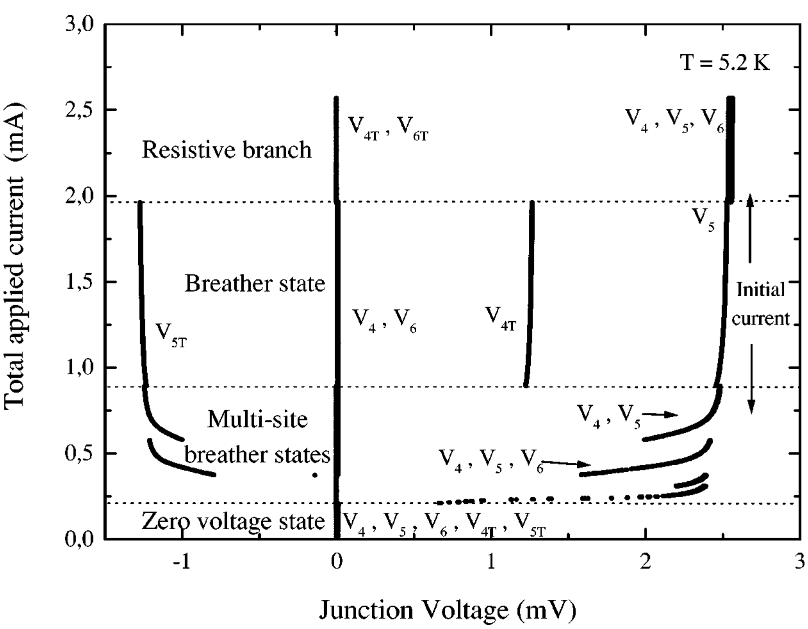

FIG. 7. Measured time-averaged voltages of five junctions in the center of the array as the applied current is varied. V_4, V_5, V_6 stand for the voltage in vertical junctions number 4, 5, and 6 in a nine junction ladder. V_{4T} and V_{5T} for horizontal junction in the top branches connecting the three previously mentioned vertical junctions. The breather state was excited at a current close to 1.4 mA. When the current was increased the breather solution became unstable at a current close to 2 mA and the array switches to a uniform resistive state, where all the vertical junctions rotate with the same voltage. We can see that for the breather state $V_4=V_6=0$ and $V_{4T}=-V_{5T}=V_5/2$ which corresponds to the breather state that we have coined as type-B state. In a second experiment, a new breather was excited but now the current was decreased. We observed that the breather becomes unstable at a current close to 0.8 mA and a new breather state is excited. This is a multi-site breather state. New instabilities between different multi-site states occur and finally the array reaches its zero-voltage state at about 0.2 mA.

D. New experiments, analysis and simulations

After the first observations of localized states in Josephson ladders, a series of new experiments and numerical work were done. Some of the main results will be reviewed here.

In Ref. 12 the Erlangen group reported on new experiments in open ladder arrays with $h=0.56$. A large diversity of rotobreather solutions were found. They detected type A, type B and other mixed solutions as well as many different multi-site breather states. They used a simple dc model to compute the retrapping current of a breather state and got a good agreement when comparing with the experiments. This model uses shorts for the librating junctions and resistances for the rotating ones and allows to include the effect of the bias resistors which help to provide a uniform bias current distribution in the ladder.

In Ref. 13 we presented an extensive study—experimental, theoretical, and numerical—of DB's in Josephson ladders. We experimentally detected a variety of rotobreather solutions (one-site, multi-site, type A, type B). The domain of existence of the solutions and the destabilization mechanisms were studied by changing the bias current and temperature. We found a nice agreement between the experiments and the predictions of the theory based on the dc circuit model. We also measured the magnetic field dependence of our solutions.

We also used the simple dc circuit model to analyze many of the experimental results. This model allows to evaluate the effect of the bias circuit and compute current–voltage curves. We also used it to estimate the maximum and minimum currents for finding DB's in the arrays. For the estimation of the minimum current, retrapping current arguments were used meanwhile the maximum current is related to junction switching mechanisms. All the predictions agree with our experimental results of small λ samples.

Numerical simulations of Eqs. (11) at the experimental values show results close to the experiments. The simulations also revealed the existence of many different DB attractors. We found periodic, quasiperiodic, and chaotic solutions. For periodic solutions, the Floquet stability analysis helped to study the destabilization of the solutions. In order to check the robustness of the rotobreather states we studied the dynamics of the solutions in the presence of thermal fluctuations incorporated into the model by adding an appropriate noise current source to each junction.

The DB also excites waves in the ladder. Using a linear analysis of the equations we calculated the frequencies of these small waves and showed that a resonance can occur when the breather frequency matches a frequency of the linear wave. We have numerically verified that these resonances can cause instabilities in the localized solutions, but in most of the cases the resonances are not strong enough for destroying the localization of the breather. We have also performed an harmonic balance approximation which allows the characterization of the amplitudes of type-A and type-B breathers.

We made extensive simulations to obtain the regions of existence of DB's for many different values of the four parameters of the system h, I, Γ, λ (Figs. 16–20 in Ref. 13). These diagrams depend importantly on the array parameters; and, in spite of the complexity, we found good agreement with the predictions of the simple model in many cases. An important observation is that in general a small Γ value results in a larger existence region. Thus, for small values of the damping ($\Gamma=0.1$ for instance) DB's are predicted to exist even at high values ($h>1$) of the anisotropy parameter. The dependence of the existence regions with λ is more complicated and only partly explained by resonances between DB frequencies and the lattice linear waves.

When varying λ we found different behaviors. At large λ, type-B solutions are strictly up–down symmetric $[\varphi_j^t(t) = -\varphi_j^b(t)]$, and at small λ this up–down symmetry is broken (although still we have a type-B solution). In the intermediate regions the resonances are important and the behavior is much more complex. It is in this region where we found chaotic localized solutions. One important conclusion of the simulations at different values of λ is the observation of resonances in the IV curves which do not destroy the DB solution. Such resonances are possible since for the existence of DB's, in the case of a forced and damped system the nonresonance condition is not necessary, because for any frequency the decay length is different from zero.

One of the more surprising experimental observations was the cascade of transitions between different multi-site breather states when decreasing the current applied to the array. Such issue was also the object of the numerical work by Giles and Kusmartsev in Ref. 18. They studied a slightly different system of equations and fixed the parameters at $h=0.44$, $\Gamma=0.07$, $\lambda=0.21$, found transition sequences similar to the experimental ones and reported on large chaotic transients in the switching processes.

The cascade of transitions and the resonances between breathers and linear modes in the ladder was also theoretically and numerically studied by Miroshnichenko *et al.*[19] Different resonances were classified and the dynamics of the tails of the breather were studied in detail. The behavior observed depends importantly on the value of λ. It was observed that in many cases the interaction between the breather and the linear modes produces voltage jumps between different breather states. In other cases the resonances produce the switching of the breather state to the superconducting branch. Recently, Schuster *et al.*[14] have experimentally observed and studied the resonances excited by the rotobreather state. The arrays were characterized by $\Gamma=0.025$, $h=0.49$ and a value of $\lambda=1.6$, higher than any of the samples previously measured. The theoretical prediction for the positions of the resonances was also compared with numerical simulations. In a different work Fistul *et al.*[15] studied the quasiperiodic dynamics found in some of the resonant breathers experimentally found in arrays with $\lambda=2.6$ and $\Gamma=0.025$.

Binder and Ustinov[16] have realized a systematic measurement of many different rotobreather states in open and closed Josephson ladder arrays. Importantly, they did the experiments in arrays with $h=1.0$ and $h=0.73$, *strongly* coupled arrays. They also discussed in detail the transitions between different states. In most of the cases the instabilities

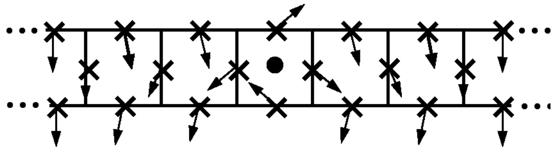

FIG. 8. A vortex configuration in the ladder ($I=0$). Arrows represent phase differences. The phases of the vertical junctions change from 0 to 2π as we move from one edge to other of the ladder.

producing such transitions were due to retrapping current mechanisms.

E. Kink–breather interaction

Another interesting question related with the dynamics of DB's is their interaction with other nonlinear excitations, such as with the vortices for the case of Josephson ladders. A static vortex in the ladder corresponds to a phase configuration similar to the one depicted in Fig. 8. In the presence of large enough external currents the static vortex is unstable and it moves along the ladder. We have numerically studied[20] the collision between one vortex and a rotobreather in the ladder. Different scenarios were found (see Fig. 9). Sometimes the rotobreather acted as a pinning center for the vortex. Then, including temperature in the model, we studied the activation process for the vortex, measured an activation energy for the vortex–breather interaction and developed a phenomenological model for estimating such energy. At higher currents the vortex excited multi-site breather states and got pinned by the breather. At higher currents a whirling mode was excited by the vortex. The breather still acts as a pinning center, now for the front. At still higher currents the traveling front excited by the vortex was able to destroy the rotobreather excitation.

F. ac biased ladders

Although all the experiments and most of the numerical work has been done in the context of a ladder biased by constant currents, the original proposal by Floría *et al.*[4] studied the excitation of localized modes in a ladder biased by an ac external field. Using a model simpler than Eqs. (11), they found oscillobreather and rotobreather localized solutions in the dynamics of the array. In both cases the solutions are periodic, with the period of the external current or some simple harmonic. These states and multi-site rotobreather solutions were also studied in Ref. 7. Martínez *et al.*[8] studied in detail the rotobreather solution. They showed results of the Floquet stability analysis of the periodic solutions done at many different values of the parameters of the array. Three main different types of bifurcations were detected and sections of the multidimensional stability phase diagram computed. In these ladders excited by ac fields, it was found and characterized a new intrinsically localized chaotic solution or *chaobreather*.[9] Such solution was obtained from a periodic one which experiences a cascade of period-doubling bifurcations when changing one of the parameters of the array.

VI. SINGLE PLAQUETTE ARRAYS

Rotobreather-like solutions can be also studied in single-plaquette arrays. Figure 10 sketches such solutions in cell arrays with three or four JJ. The case of a square cell with four JJ was considered in Refs. 11 and 13. In this case the solution can be seen as a reduction of the solutions found in the ladder where if we impose mirror symmetry with respect to the rotating junction of a one-site rotobreather and, thanks to the localization of the breather solution, neglect the dynamics of junctions beyond the first neighbor to the rotating, we are left with a square plaquette with four junctions. The equations for the plaquette can be mapped onto the equation of the ladder with $h_{\mathrm{plaq}}=2h_{\mathrm{ladd}}$ and $\lambda_{\mathrm{plaq}}=2\lambda_{\mathrm{ladd}}$. Then it was found that many of the main aspects concerning DB solutions in the ladder can be studied in single cell arrays (see Fig. 5 in Ref. 11 and Figs. 25 and 27 in Ref. 13).

Benabdallah *et al.*[21] have done an analytical and numerical study of the rotobreather states in a single plaquette with three junctions. They showed that this system is complex enough to show most of the phenomena observed in Josephson ladders. They realized an exhaustive computation of the regions of existence of the solutions as a function of the array parameters and found that the rotobreather states exist in a large range of control parameters. They also studied the stability and the resonant behavior of the breather solution and found that resonances are very important for moderate values of λ, where large instability regions appear. Such instabilities are very sensitive to the damping parameter.

Fistul *et al.*[22] studied the effect of external magnetic fields in the dynamics of rotobreather states in a single cell with three junctions. They analyzed mainly the effect of the field on the breather resonances and instabilities, and showed that using the applied field it is possible to control such instabilities. In particular they show how to increase or decrease the height of the resonant steps and to suppress the voltage jumps in the current–voltage characteristics.

The single loop with three junctions has been experimentally studied by Pignatelli and Ustinov[23] very recently. They were able to measure the solutions in a wide range of parameter values. The study of the range of existence of the localized solutions confirmed the theoretical predictions based on the application of the dc model to the plaquette.[21] They also studied the behavior of the different resonances observed in the plaquette and the effect of external magnetic fields. The experimental results are in a very good agreement with the previous analytical predictions of Ref. 22.

VII. TWO-DIMENSIONAL JJ ARRAYS

Josephson junctions can be also connected making a two-dimensional array. Such arrays have been paradigmatic experimental systems for the study of many dynamical and thermodynamical problems: the Kosterlitz–Thouless transition, commensurability effects, vortex flow, synchronization, etc.[24]

It also is worth mentioning that complex inhomogeneous resistive patterns in dc-driven 2D arrays have been experi-

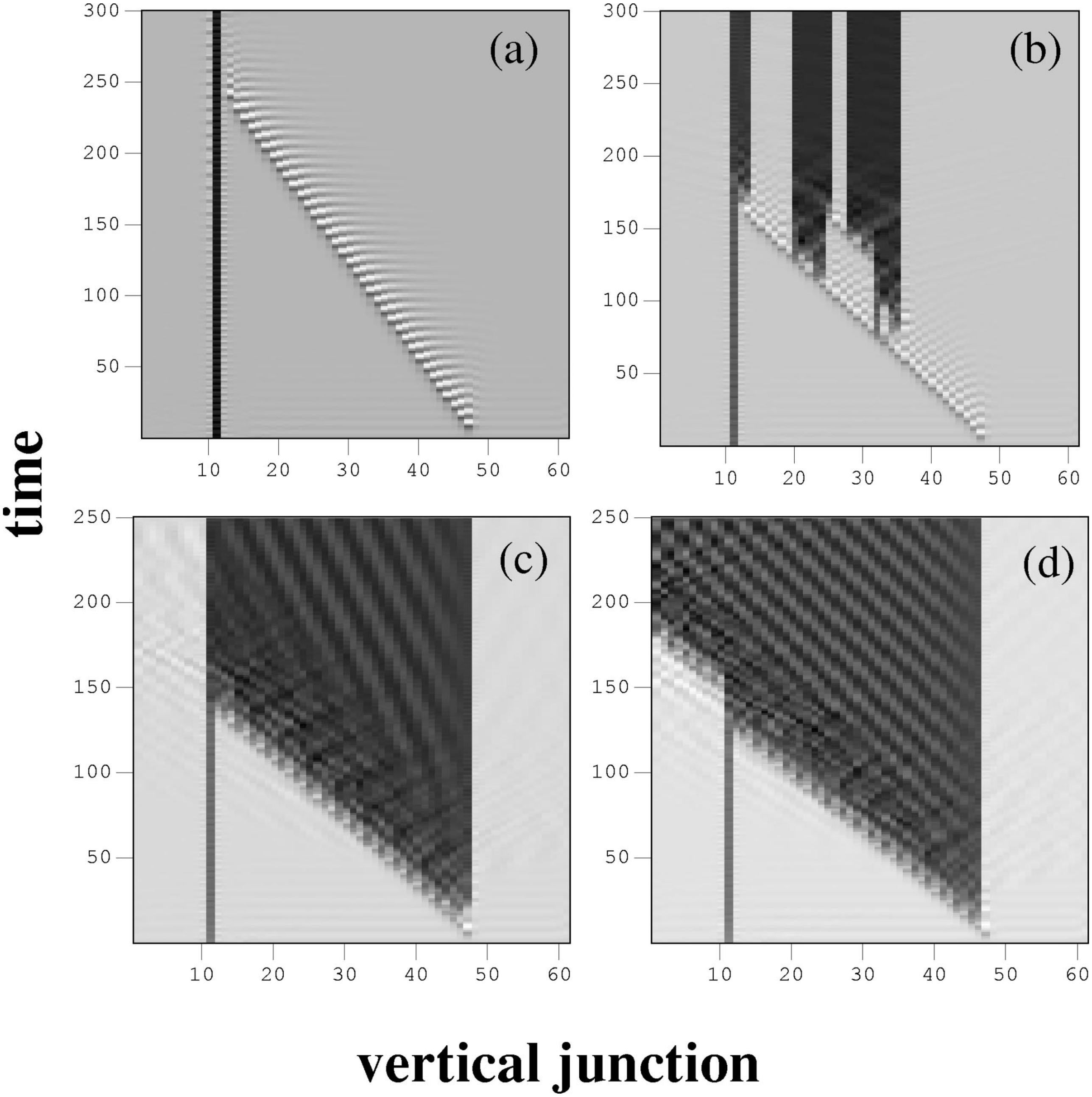

FIG. 9. Simulations of vortex–breather collision in a Josephson ladder. For all the cases the initial condition is a one-site rotobreather in junction 11 of the ladder and a vortex in junction 47. The figures plot the value of the instantaneous voltage at every junction. $\Gamma=0.1$, $f^{\text{ext}}=0.3$, $h=0.5$, and $\lambda=5.0$. From (a) to (d) we have increased the applied current. (a) $i=0.45$, the moving vortex is pinned by the breather. (b) $i=0.5$ the vortex switches some of the vertical junctions to rotate. The final state shows three multi-site rotobreathers with 3, 6, and 8 vertical rotating junctions. (c) $i=0.55$, the vortex excites all of the vertical junctions in its wake. In the final state the ladder is divided in three domains, two superconducting and one resistive in the center. (d) $i=0.59$, the vortex switches vertical junctions and annihilate the breather.

mentally observed by Abraimov *et al.*[25] Such patterns correspond to a broken symmetry of row switching in the array.

The existence of DB's in two two-dimensional JJ arrays (2DJJA) has been studied recently in Ref. 26. Based on theoretical arguments from the uncoupled limit of the lattice and on numerical simulations, rotobreather solutions have been predicted to exist in 2DJJA biased by ac currents. Such localized solutions correspond to the voltage localized solution depicted in Fig. 11: four junctions sited in the bulk of the array are in the resistive state, two with voltages $+V$ and two

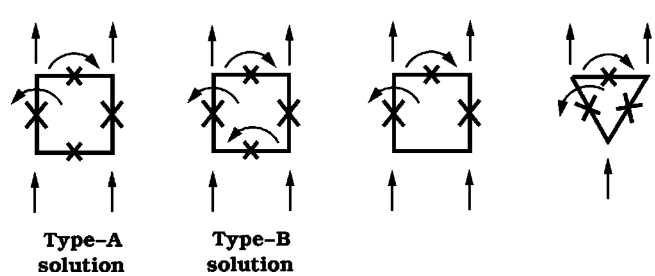

FIG. 10. Sketches of generalized breather solutions in single cell arrays with four and three Josephson junctions.

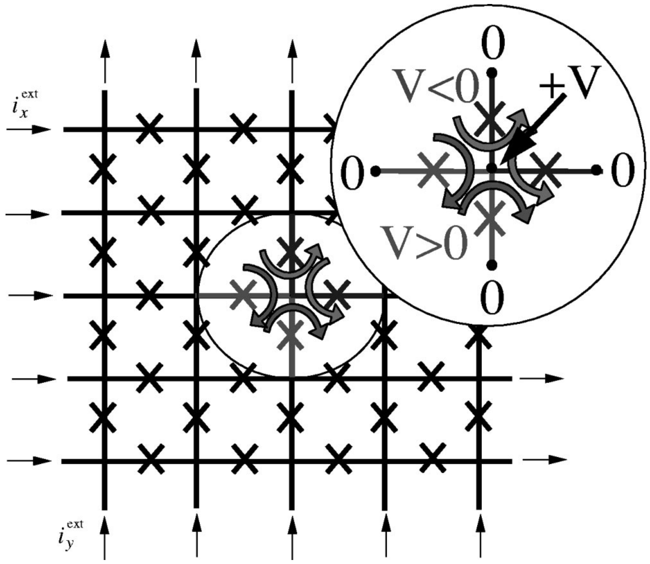

FIG. 11. Sketch of the 2DJJA with breather.

with $-V$, while the rest follow the ac field in an oscillating state of mean voltage $V=0$.

The breathers have been numerically obtained in many different situations. The possibility of parallel ($i_x^{ext}=0$, $i_y^{ext}\neq 0$) or diagonal ($i_x^{ext}=i_y^{ext}\neq 0$) bias were considered. From simulations with single junctions we mostly studied breathers with $\Gamma=0.02$ and $\omega=\pi/2$ [from Fig. 2(b)], and $\Gamma=0.003$ and $\omega=\pi$ [from Fig. 2(c)]. With respect to the coupling parameters λ and h, different types of rotobreather solutions were obtained either at $h=1$ and small λ; $h=1$ and very large λ or small h. Most of the rotobreathers resulted to be periodic with the period of the external field and in addition to standard numerical simulations of the dynamics we studied the stability of the solutions by means of Floquet analysis and the robustness of the solutions with respect to external noise by introducing noise current sources in the dynamics of the junctions. Simulations were done in 11×11, 25×25, and 100×100 arrays.

Figure 12 shows four snapshots along one period of the phase dynamics for two rotobreather solutions simulated in two very different situations (see captions).

In relation to the issue of a possible experimental observation of DB's in 2DJJA, the simulations were done at experimentally accessible parameter values and it was found a numerical protocol to excite DB's in the array. The protocol is based in the possibility of adding a local dc current that should be injected in one central island of the array and extracted from the four neighboring islands: First, using the local dc injection it is possible to cause four junctions to rotate, once this occurs we decrease the current to a small value but large enough to maintain the localized rotation. Then the ac field is applied increasing slowly the amplitude of the signal from zero. The process ends by eliminating the remaining small local dc bias. With respect to the detection, it can be done for instance by measuring local voltages in different points of the array.

VIII. CONCLUSIONS

In the last six years an important effort has been done to study and observe DB solutions in Josephson arrays. Most of this effort was concentrated in Josephson ladder arrays and dc biased breathers. The main issues concerning the dynamics of the modes have been studied and satisfactorily understood. Recent experiments have been also done in dc biased single cells.

However, there are still interesting questions to answer

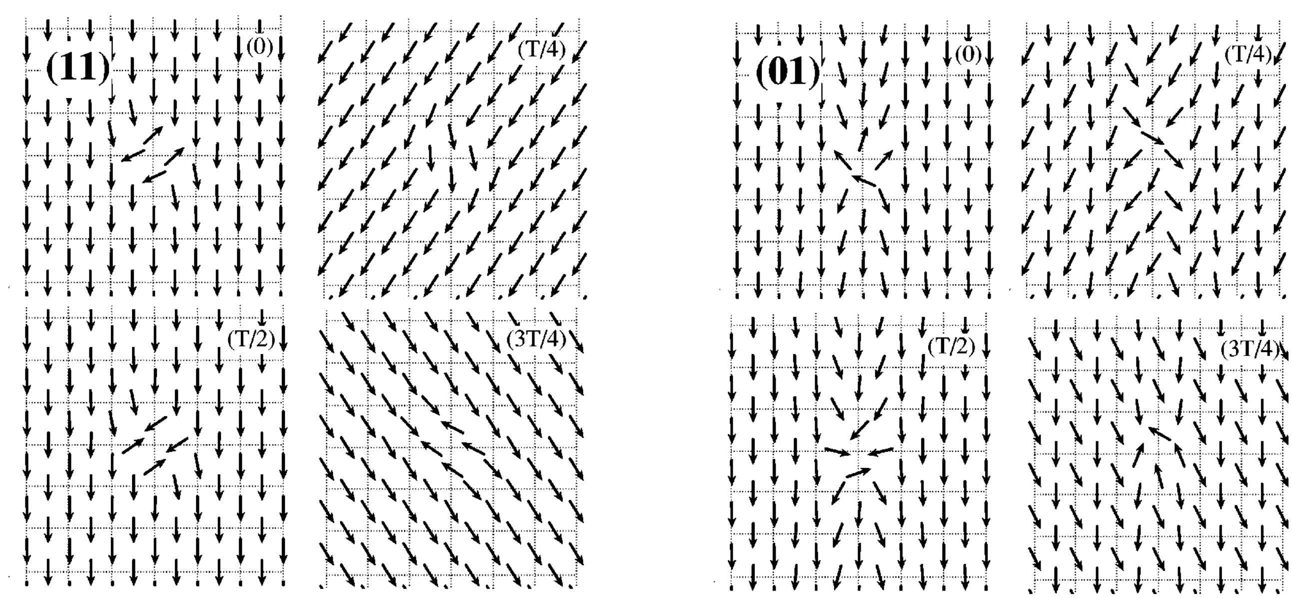

FIG. 12. Four snapshots of the phase evolution for two different DB solutions in a 2DJJA with different bias and very different parameter values. Arrows represent phases of the junctions and we show the central part of a 11×11 array. The solutions are time periodic (period T) and time increases from 0 (arbitrary) to $3T/4$ as labeled. The rotobreather solution is localized in the four central junctions which rotate. Left, current is biased in the diagonal or (11) direction ($\lambda=0.1$, $h=1.0$, $i_{ac}=5.0$, $\Gamma=0.003$, and $\omega=\pi$). Right, current is biased along the y or (01) direction ($\lambda=5.0$, $h=0.05$, $i_{ac}=0.7$, $\Gamma=0.02$, and $\omega=\pi/2$).

and new experiments to design. The excitation and observation of ac biased breathers is perhaps the most interesting one. Such work can be done in single cells, ladders and two-dimensional arrays, in order of complexity.

Other of the aspects which still lack of an experimental study is the interaction of DB's with vortices in Josephson arrays.

We finish with another interesting issue to consider, as it is the role, if any, of localized modes in the general problem of approach to equilibrium in Josephson arrays.

ACKNOWLEDGMENTS

The authors thank F. Falo, M. V. Fistul, S. Flach, L. M. Floría, J. L. García-Palacios, P. J. Martínez, F. Pignatelli, M. Schuster, K. Segall, E. Trías, and A. V. Ustinov for insightful discussions. Financial support is acknowledged to DGES PB98-1592, FEDER Program through Project No. BFM2002-00113, European Network LOCNET HPRN-CT-1999-00163, the Commission for Cultural, Educational and Scientific Exchange between the United States of America and Spain and NSF Grant No. DMR-9988832.

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