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Josephson-junction ladder: A benchmark for nonlinear concepts

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Abstract

The theoretical analysis of the ground state properties and dissipative dynamics of an anisotropic ladder of Josephson junctions has revealed interesting features associated to the nonlinear character of the Josephson effect, combined with the inherent discreteness of the system and the peculiarities of the ladder geometry. We analyse some aspects of its underdamped dynamics when spatially homogeneous time-periodic currents are injected into the islands, and predict the existence of attracting time-periodic spatially localised modes, for some ranges of junction characteristic parameters. These elementary dynamical excitations are of two different types, associated to oscillatory and rotating motion of a few superconducting island phases, respectively, revealing a dynamical mechanism of creation of vortex–antivortex pairs. These results are physical applications of recent advances in the theory of nonlinear dynamics of discrete macroscopic systems. Their experimental confirmation would probe the physical relevance of localisation in superconducting devices. Copyright © 1998 Elsevier Science B.V.

1. Introduction

The impressive developments of nonlinear science in the last decades have provided new concepts and powerful perspectives which are penetrating many areas of physical research, shedding new light and raising new questions about an increasing number of physical systems. Condensed matter physics is not an exception in this respect, as important contributions from nonlinear approaches to several condensed matter systems have been well-known for years [1]. In particular, Josephson-junction devices constitute an active research field of both fundamental and practical importance, which has received a great deal of attention from the nonlinear side. The theory of the Josephson effect contains essentially nonlinear aspects and its predictions can be seen to be closely related to the general physics of a forced and damped mathematical pendulum, one of the simplest model systems exhibiting a very complex dynamics.

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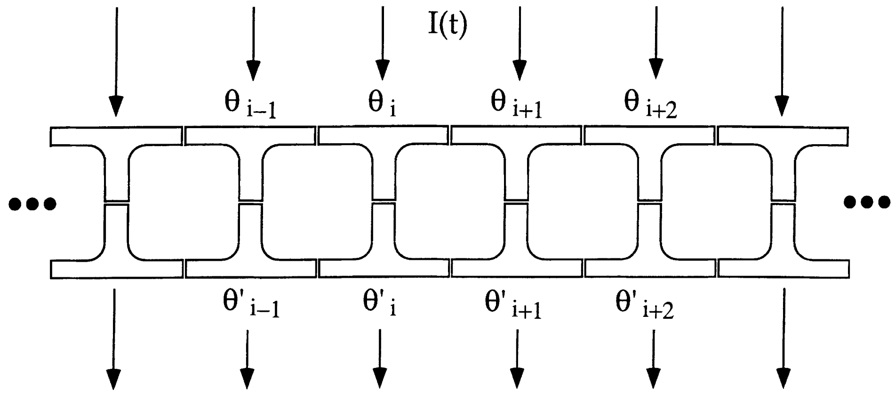

Fig. 1. Schematic picture of the Josephson-junction ladder showing the injection of the currents in the array.

We will consider here a Josephson-junction array with a ladder geometry (see Fig. 1). This type of geometry presents some specific features which distinguish it from both purely one- and two-dimensional geometries, specially when anisotropy in the horizontal and vertical directions is present.

The anisotropic Josephson-junction ladder was introduced by Kardar [2], by suggestion of Halsey, in connection with the equilibrium properties of the discrete sine-Gordon (standard Frenkel–Kontorova) model; both models are different, but in the strong coupling limit, the latter is a good approximation to the former. More recently, Mazo et al. [3] have shown (with no approximations) that the ground state problem of the Josephson-junction ladder under perpendicular magnetic field exhibits the same characteristic features as that of the standard Frenkel–Kontorova model. This result is based upon certain equivalence theorems due to Sasaki and Griffiths [4], and has the consequence that the equation of state (fluxoid quanta density versus magnetic field) is a devil's staircase, which changes from incomplete to complete when the anisotropy parameter crosses a critical value (Aubry's breaking of analiticity) [5]; at an introductory level, see [6]. The consideration of inductive effects [7] does not modify those ground state properties.

The equivalence between the ground state problems of the Josephson-junction ladder and the standard Frenkel–Kontorova model does *not* extend to their dissipative dynamics. Though both systems share a number of qualitative dynamical features concerning depinning transition, mode-locking under time-periodic forces [8] or glassy-like relaxation toward equilibrium, the underlying differences manifest often in an important way [6]. The same applies to the inertial dynamics, though much remains to be studied in this respect for both systems.

We have recently analysed the dynamics of a Josephson-junction ladder with injected AC currents, searching for intrinsic localised modes (also known as discrete breathers). These types of time-periodic solutions are attractors of the dynamics of the ladder in a wide range of parameters. They are also robust against small stochastic noise. They can be classified into two groups: (i) Oscillator localised modes, in which the amplitude of the superconducting phase oscillation is exponentially localised, and (ii) rotor localised modes, where the phases of a pair of superconducting islands rotate while the rest perform forced oscillations. Their physical characterization reveals some interesting features which could be of relevance in order to their eventual experimental observation. Before reporting on these results we discuss, in Section 2, the physical description of the Josephson-junction ladder and derive the resistively and capacitively shunted junction (RCSJ) approach to its dynamics. A preliminary report on these results has appeared in [9].

We would like to echo the words of Robert Mackay in this conference, calling for experiments to probe recent theoretical advances in nonlinear intrinsic localisation, and push forward the Josephson-junction ladder as a benchmark system where the necessary interaction between theory and experiments can take place.

2. Josephson-junction arrays and RCSJ approach

Superconducting arrays consist of superconducting grains (islands) embedded in a normal or insulating matrix, linked together by Josephson or proximity effect couplings. The superconducting Ginzburg–Landau order parameter (many-particle condensate wave function) of the island located at position $\boldsymbol{x}$ is denoted by

$$\Psi(\boldsymbol{x}) = (N(\boldsymbol{x}))^{1/2}\exp(\mathrm{i}\theta(\boldsymbol{x})), \tag{1}$$

where $N(\boldsymbol{x}) = |\Psi(\boldsymbol{x})|^2$ is the local density of superconducting pairs (charged bosons) and $\theta(\boldsymbol{x})$, the phase, its conjugate variable.

For the ideal case of perfect insulating junctions (i.e. no ohmic currents), the Josephson Hamiltonian of a homogeneous array can be written as

$$H_\mathrm{J} = \sum_{\boldsymbol{x}\in\Lambda}\left[-\mu|\Psi(\boldsymbol{x})|^2 + \lambda|\Psi(\boldsymbol{x})|^4 - \sum_{\boldsymbol{\delta}} T_{\boldsymbol{\delta}}\Psi^*(\boldsymbol{x})\Psi(\boldsymbol{x}+\boldsymbol{\delta})\right], \tag{2}$$

where Λ is the lattice of island positions, $\boldsymbol{\delta}$ are nearest neighbour lattice vectors, μ is the chemical potential, λ is proportional to the island inverse capacitance and $T_{\boldsymbol{\delta}}$ is the coupling of the junction attached to the link $\boldsymbol{\delta}$.

The Schrödinger equation for the island condensate wave function $\Psi(\boldsymbol{x})$ is

$$\mathrm{i}\hbar\dot{\Psi}(\boldsymbol{x}) = \frac{\delta H_J}{\delta\Psi^*(\boldsymbol{x})} = -\mu\Psi(\boldsymbol{x}) + 2\lambda|\Psi(\boldsymbol{x})|^2\Psi(\boldsymbol{x}) - \sum_{\boldsymbol{\delta}} T_{\boldsymbol{\delta}}\Psi(\boldsymbol{x}+\boldsymbol{\delta}), \tag{3}$$

where one easily recognises a *discrete nonlinear Schrödinger equation*. This equation can be reexpressed in the form of equations of motion for the particle number $N(\boldsymbol{x})$ and phase $\theta(\boldsymbol{x})$, by inserting Eq. (1), multiplying both sides by $\Psi^*(\boldsymbol{x})$ and separating real and imaginary parts:

$$\dot{N}(\boldsymbol{x}) = -\frac{2}{\hbar}\sum_{\boldsymbol{\delta}} T_{\boldsymbol{\delta}}[N(\boldsymbol{x}+\boldsymbol{\delta})N(\boldsymbol{x})]^{1/2}\sin[\theta(\boldsymbol{x}+\boldsymbol{\delta}) - \theta(\boldsymbol{x})], \tag{4}$$

$$\dot{\theta}(\boldsymbol{x}) = \frac{1}{\hbar}\left[\sum_{\boldsymbol{\delta}} T_{\boldsymbol{\delta}}\left[\frac{N(\boldsymbol{x}+\boldsymbol{\delta})}{N(\boldsymbol{x})}\right]^{1/2}\cos[\theta(\boldsymbol{x}+\boldsymbol{\delta}) - \theta(\boldsymbol{x})] + \mu - 2\lambda N(\boldsymbol{x})\right]. \tag{5}$$

If it is the case that the superconducting islands are of macroscopic size, then the *Josephson approximation*

$$T_{\boldsymbol{\delta}} \ll 2\lambda N(\boldsymbol{x}) \tag{6}$$

should be valid so that Eqs. (4) and (5) reduce to

$$\ddot{\theta}(\boldsymbol{x}) = \sum_{\boldsymbol{\delta}}[J_{\boldsymbol{\delta}}\sin(\theta(\boldsymbol{x}+\boldsymbol{\delta}) - \theta(\boldsymbol{x}))], \tag{7}$$

where $J_{\boldsymbol{\delta}}$ is the critical current, $I_{\mathrm{c},\boldsymbol{\delta}}$, of the junction attached to the link $\boldsymbol{\delta}$. When resistive effects associated to the contribution of normal electrons are included in this description, and external currents $I(t)$ are injected into each island, the equations of motion for the Josephson-junction array (neglecting inductive effects) are

$$\ddot{\theta}(\boldsymbol{x}) = \sum_{\boldsymbol{\delta}}[J_{\boldsymbol{\delta}}\sin(\theta(\boldsymbol{x}+\boldsymbol{\delta}) - \theta(\boldsymbol{x})) + \epsilon_{\boldsymbol{\delta}}(\dot{\theta}(\boldsymbol{x}+\boldsymbol{\delta}) - \dot{\theta}(\boldsymbol{x}))] - I(t). \tag{8}$$

These equations of motion for the array are known as the RCSJ model [10,11], which is usually considered to give a sensible description for the so-called *classical junction* regime, in which the ratio between charging and Josephson energies is very small (see discussion in Section 5).

Let us concentrate now on the particular geometry of the ladder as shown in Fig. 1. We will assume that the junction characteristics are (J_x, ϵ_x) for junctions in the horizontal rows, and (J_y, ϵ_y) for junctions which couple islands in different rows. Also, θ_i and θ_i' will denote, respectively, the phases of upper and lower islands at site i in the ladder. Moreover, the currents $I(t)$ are injected into the islands in the upper row and extracted from those in the lower row. Then, from Eq. (8),

$$\begin{aligned}\ddot{\theta}_i &= J_x[\sin(\theta_{i+1}-\theta_i)+\sin(\theta_{i-1}-\theta_i)] + J_y\sin(\theta_i'-\theta_i)\\ &\quad + \epsilon_x(\dot{\theta}_{i+1}+\dot{\theta}_{i-1}-2\dot{\theta}_i)+\epsilon_y(\dot{\theta}_i'-\dot{\theta}_i) - I(t),\\ \ddot{\theta}'_i &= J_x[\sin(\theta'_{i+1}-\theta'_i)+\sin(\theta'_{i-1}-\theta'_i)] + J_y\sin(\theta_i-\theta'_i)\\ &\quad + \epsilon_x(\dot{\theta}'_{i+1}+\dot{\theta}'_{i-1}-2\dot{\theta}'_i)+\epsilon_y(\dot{\theta}_i-\dot{\theta}'_i) + I(t).\end{aligned} \tag{9}$$

With the change of variables (centre of mass and relative coordinates) $\chi_i = \frac{1}{2}(\theta_i+\theta_i')$, $\phi_i = \frac{1}{2}(\theta_i-\theta_i')$, Eqs. (9) can be written as

$$\ddot{\chi}_i = J_x[\sin(\chi_{i+1}-\chi_i)\cos(\phi_{i+1}-\phi_i)+\sin(\chi_{i-1}-\chi_i)\cos(\phi_{i-1}-\phi_i)] + \epsilon_x(\dot{\chi}_{i+1}+\dot{\chi}_{i-1}-2\dot{\chi}_i), \tag{10}$$

$$\begin{aligned}\ddot{\phi}_i &= J_x[\cos(\chi_{i+1}-\chi_i)\sin(\phi_{i+1}-\phi_i)+\cos(\chi_{i-1}-\chi_i)\sin(\phi_{i-1}-\phi_i)]\\ &\quad + \epsilon_x(\dot{\phi}_{i+1}+\dot{\phi}_{i-1}-2\dot{\phi}_i) - J_y\sin(2\phi_i) - 2\epsilon_y\dot{\phi}_i - I(t).\end{aligned} \tag{11}$$

With uniform initial conditions in the centre of mass coordinates and momenta, i.e., χ_i, $\dot{\chi}_i$ independent of i, (10) have the solutions $\chi_i(t) = \Omega t + \alpha$ for all i. In this case, the dynamics reduces to the following equations of motion [1] for the phase half-differences, ϕ_i:

$$\ddot{\phi}_i = \tilde{\eta}[\sin(\phi_{i+1}-\phi_i)+\sin(\phi_{i-1}-\phi_i)] + \tilde{\epsilon}_x(\dot{\phi}_{i+1}+\dot{\phi}_{i-1}-2\dot{\phi}_i) - \sin(2\phi_i) - 2\tilde{\epsilon}_y\dot{\phi}_i - \tilde{I}(t), \tag{12}$$

where, from now on, we will assume $\tilde{I}(t) = I\cos(\omega t)$ for the injected currents. These are the equations of motion for an array of forced and damped nonlinear rotors with periodic (sinusoidal) coupling between nearest neighbours and phonon dissipation. It is important to realise that the sinusoidal character of the coupling allows the rotation of individual rotors, provided the parameters $\tilde{\eta}$ and $\tilde{\epsilon}_x$ are not too large. This possibility is ruled out in the continuum approximation, *only* valid in the strong coupling limit. *Discreteness* and *periodicity* of the couplings are essential features of the Josephson-junction ladder, which together make important differences respect to other superconducting devices like long Josephson junctions or some circular arrays which have been recently studied [12].

3. Discrete breathers and numerical procedure

The existence and characterisation of intrinsically localised modes in Hamiltonian discrete arrays of nonlinear oscillators has been anticipated by different analytical and numerical approximations [13–15] and has recently been given support by new theorems of nonlinear dynamical systems theory [16] exploiting the ideas of anti-continuous limit [17,18]. Many issues concerning their stability, movability, etc. are still open questions in spite of recent progress [19–21]. These localised modes (also known as *discrete breathers*) are intrinsic, in the sense that they occur even if the system is homogeneous (no impurities or disorder is present), so that the localisation is due to nonlinearity.

[1] The unit time from Eq. (12) onwards is $\tau = \hbar(8E_C E_J)^{-1/2}$. The dimensionless anisotropy parameter is $\tilde{\eta} = J_x/J_y = I_{c,x}/I_{c,y}$, and the dissipative couplings $(\alpha = x, y)$ $\tilde{\epsilon}_\alpha = (\hbar/e^2)R^{-1}(E_C/E_{J_\alpha})^{1/2}$; here $E_C = e^2/(2C)$ and $E_J = \hbar I_c/(2e)$ are the charging and Josephson energies respectively. Finally, $\tilde{I}(t)$ is measured in units of $I_{c,y}$.

It has been already argued in [19] (see also [22]), in a general way, that by adding both dissipation and a spatially uniform time-periodic driving force to a Hamiltonian array of nonlinear oscillators, the intrinsic localised modes corresponding to the driving frequency not only persist, but many of them become attractors for the dynamics of the whole coupled system. The equations of motion (12) for the Josephson-junction ladder are in this sense a physical example of such a type of many-degrees-of-freedom dynamical systems where intrinsic localised modes are solutions with an open basin of attraction in phase space.

In order to compute explicitly these localised solutions we have used the numerical procedure developed in [19]. Based on rigorous mathematical foundations and incorporating well-established numerical methods of fixed point iteration and integration of ODEs, this general method provides the exponentially localised solution as a fixed point of a functional operator in certain space of sequences of functions. In our case, this is the space of loops $\bar{\phi} = \{\phi_i(t), \dot{\phi}_i(t)\}$, $i = -N, \ldots, N$, with $\phi_i(t + 2\pi/\omega) = \phi_i(t)$, and $\dot{\phi}_i(t + 2\pi/\omega) = \dot{\phi}_i(t)$, where ϕ is an angular variable. The operator is the Poincaré (stroboscopic) map $\mathcal{T}_{t_b}$, which maps the initial conditions $\{\phi_i(0), \dot{\phi}_i(0)\}$ to $\{\phi_i(t_b), \dot{\phi}_i(t_b)\}$, where $t_b = 2\pi/\omega$, and we use a Newton method to compute the fixed point. Starting from the uncoupled array ($\tilde{\eta} = \tilde{\epsilon}_x = 0$), where the localised solution is trivial, and varying the couplings by small steps, the Newton method is very efficient for finding recursively the intrinsic localised solution for finite couplings, provided that the solution obtained in the previous step is taken as the initial trial solution. By this iterative process, we follow "quasi-continuously" the discrete breather from the uncoupled limit. As a byproduct of the method [19], the eigenvalues of the linear tangent map $\partial_{\bar{\phi}}\mathcal{T}_{t_b}$, used by the Newton method, provides us the linear stability analysis of the intrinsic localised mode, which allows the analysis of the mechanisms which destabilise the localisation.

The solution obtained with this procedure can then be given as the initial condition to a direct simulation of the full system of equations of motion (10) and (11); remember that Eqs. (12) correspond to a particular choice of initial conditions for the "centre of mass" variables, which allows the decoupling of χ_i and ϕ_i variables. That simulation allows us to check the unrestricted linear stability of the intrinsic localised modes, their attracting character in full phase space, robustness against different types of fluctuations, etc.

Although we are not aware of a rigorous proof, it may be that intrinsic localized modes in forced-damped continuous nonlinear systems are structurally stable, as suggested in [23]. In any case, note that a "rotobreather" cannot exist in a continuous system, so that discreteness is a prerequisite for, at least, a good part of our results.

4. Results

Fig. 2 shows the energy profile of an oscillator localised mode in the ladder for a particular choice of the parameter values. There we also show the closed trajectories in the phase space (ϕ, $\dot{\phi}$) of the phase half-differences across vertical junctions. The high precision of the numerical method of Marín and Aubry allows for a precise Floquet analysis of the linear stability of these solutions. We are currently carrying out an exhaustive exploration in parameter space of this issue, in order to determine the instability mechanisms. From our preliminary results we certainly conclude the existence of wide zones for linear stability of the oscillator localised modes, mostly for low values of the anisotropy $\tilde{\eta}$. This is what should be expected, for localisation is naturally favoured by low values of the horizontal coupling between superconducting phases.

The second class of localised solutions, the rotor localised mode, differs from the previous one in that at the site of energy localisation, the superconducting phases rotate, while the rest remain oscillating (see Figs. 3 and 4). For a Hamiltonian array of pendula with periodic coupling, this type of localised mode was first found explicitly by Takeno and Peyrard [24], and its existence was implicitly envisaged in the theorems of [16]. Note that in the Josephson-junction ladder there are additional external forcing, damping and momenta coupling. In this situation, the *rotobreather* becomes asymptotic steady-state for a positive measure open basin of attraction in full phase space.

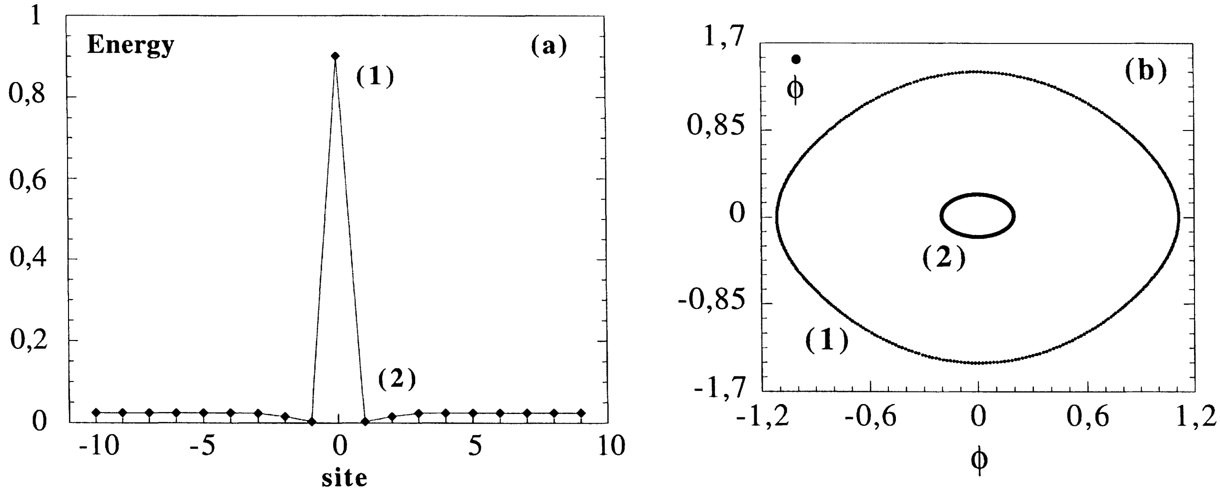

Fig. 2. (a) Average energy (kinetic + Josephson) profile of an oscillator localised mode ($J_y = 0.5$, $J_x = 0.04$, $\epsilon_y = \epsilon_x = 0.01$, $I = 0.1$, $\omega = 0.8$). (b) Phase space trajectories of the central (1) and its nearest neighbor (2) phases.

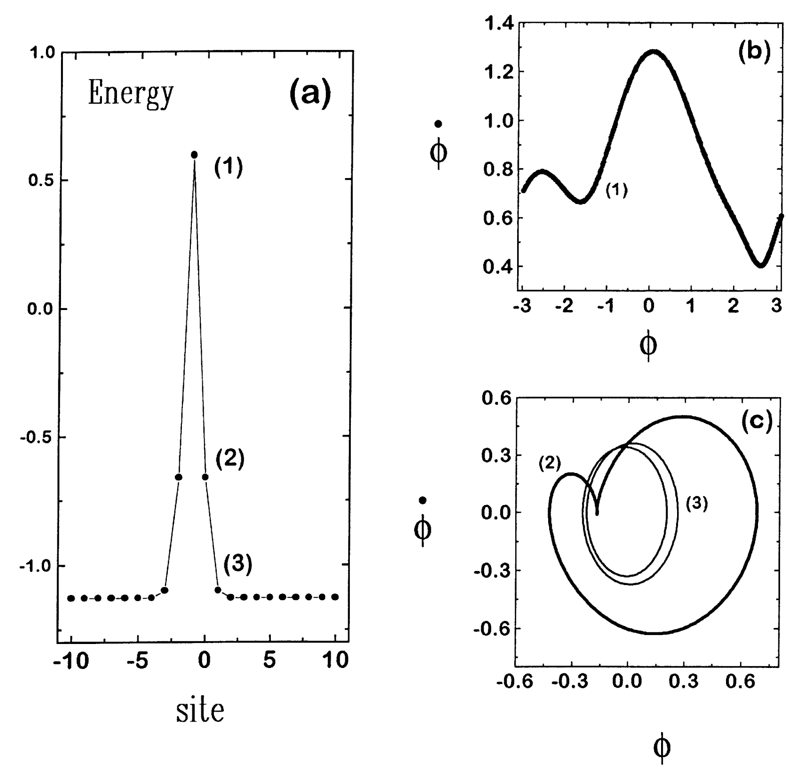

Fig. 3. (a) Average energy profile of a rotor localised mode for the parameter values $J_y = 0.5$, $J_x = 0.18$, $\epsilon_x = \epsilon_y = 0.01$, $I = 0.3$, $\omega = 1.5$. (b) and (c) Phase space trajectories for the central (1) phase of the discrete rotobreather and its nearest (2) and next nearest (3) neighbors.

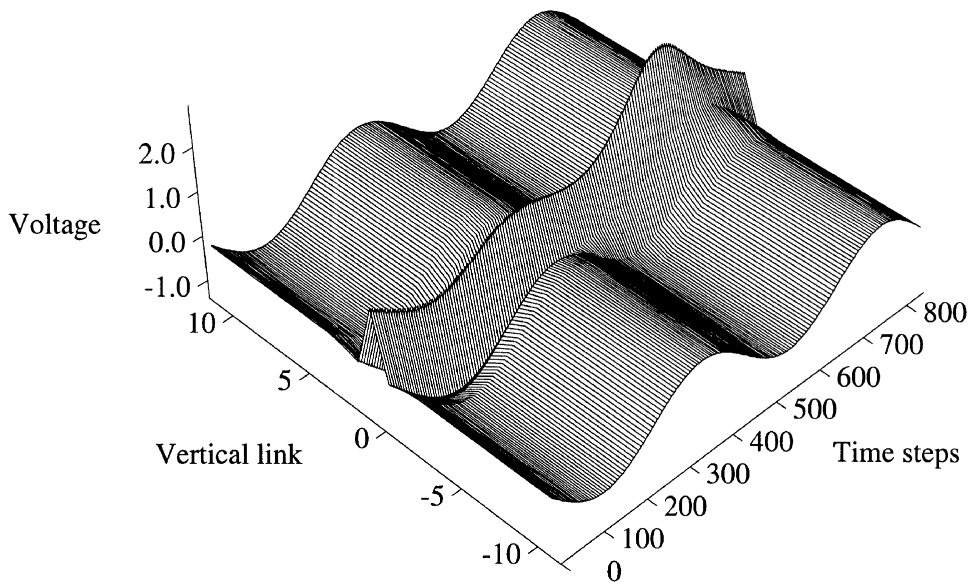

Fig. 4. Voltage profiles (in units of $\hbar/(2e\tau)$) across vertical links, for a rotobreather solution. The integration time step in this and Fig. 5 is 0.01.

We have checked the attracting character of these solutions (both oscillator and rotor localised modes) by direct numerical integration of the original equations of motion (10) and (11). For arbitrary, small enough, perturbations of the localised solution, the perturbed trajectories are observed to tend to the unperturbed localised mode, confirming that it is an attractor of the dynamics. They are also robust under small stochastic additive noise and quenched disorder in the junction characteristics. This clearly suggests a non-negligible chance of observing these localised modes in experiments, for an appropriate range of fabricated junction characteristics.

In analogy with [16] we predict also the existence of arrays of modes localised at different sites. We have computed also these multi-mode solutions and confirmed their attracting character (see Fig. 5). In fact, in the range of low values of (J_x, ϵ_x), for a wide set of arbitrary initial conditions, the attractor of the full dynamics in phase space which is reached after (sometimes longer, often shorter) transients is generically a (glassy-like) array of localised modes.

One of the most significant features of a rotor localised mode concerns the vorticity profile of the ladder. The number of fluxoid quanta (vortices) n_i associated to the ith plaquette in the ladder is defined in terms of the circulation around the plaquette of the superconducting phase gradient. In the absence of external magnetic field, which is the case we analyse here, one has

$$\oint_{\Gamma_i} \nabla\theta \cdot \boldsymbol{d}l = 2\pi n_i, \tag{13}$$

where the contour integral is performed around a loop Γ_i enclosing the ith plaquette. Fig. 6 shows the time-dependent vorticity profile of the ladder for a rotor localised mode computed with the method of Marín and Aubry. The excitation energy of this mode (which is a microscopic quantity due to the localised character of the solution) manifests itself through the intermittent creation and subsequent annihilation of a pair of fluxoid quanta of opposite sign (vortex–antivortex) located in the central plaquettes.

But what also makes the rotor localised mode very interesting is its associated voltage profile. As can be seen in the example of Fig. 4, the voltage through the vertical link where localisation occurs has non-zero mean value, so

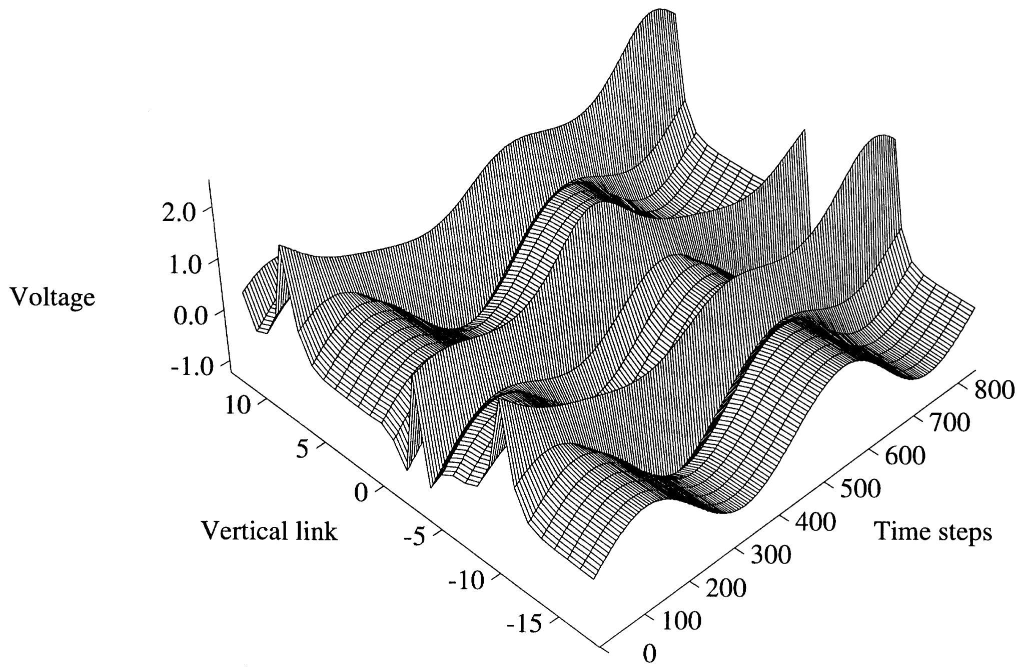

Fig. 5. Voltage profiles across vertical links for a multi-rotobreather solution.

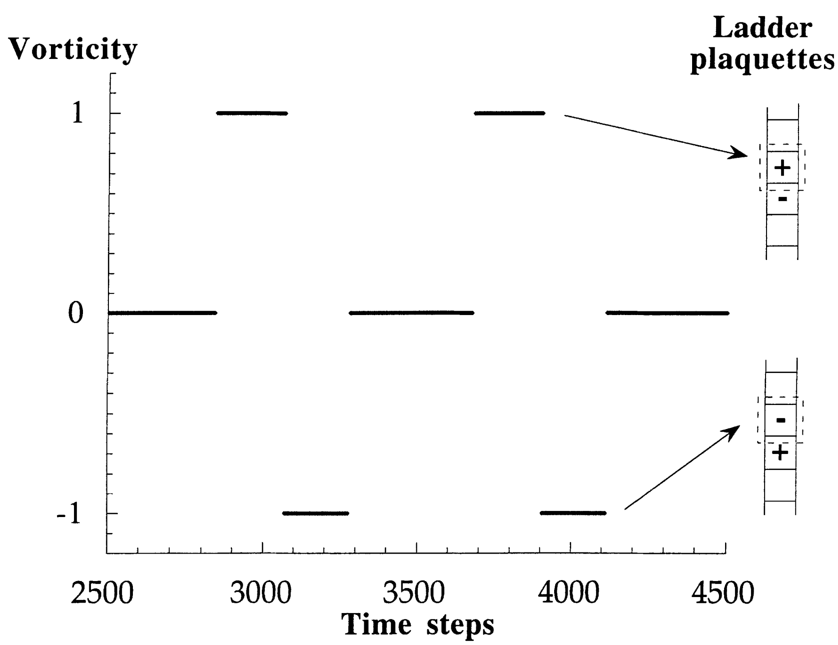

Fig. 6. Time evolution of the vorticity of one of the central plaquettes for the rotor localised mode of Fig. 3.

that a *direct measure* of it will reveal the presence (absence) of the *rotobreather* in a feasible experiment, provided the device is adequately designed.

5. Concluding remarks

It is plain that many issues concerning intrinsic localised modes in the Josephson-junction ladder are open questions.

(i) The comprehensive characterisation of the mechanisms responsible for the destabilisation of these solutions, along with their movability, are some of the issues under current study.

(ii) Although we have considered the situation in which no external magnetic field is present, it is not hard to realise that the existence of energy localisation in the Josephson-junction ladder persists under magnetic fields. Interesting phenomena are then expected to appear as a result of the interplay between commensurability (and the associated defectibility [6]) and localisation. For example, how the pressence of an intrinsic localised mode of either type influences vortex motion?

(iii) What is the role of these localised modes in the energy transport properties of the ladder? In particular, how do they relate to intrinsic disorder?

(iv) In any case, we should recall that most applications of Josephson-junction array based devices rely on the coherent synchronous motion of the phases; what we have shown here constitutes a new phenomenon which could destroy this coherence (through the eventual unbinding of pairs and subsequent vortex motion), and is therefore something to be avoided in such devices, or perhaps, due to the robustness and generality of this novel effect, something to be turned to our advantage in possible new applications (e.g., photodetection).

This enumeration is by no means exhaustive, but could serve to support the interest in experimental studies of the Josephson-junction ladder, where specific questions ranging from the several aspects on the design of the superconducting device, to the conditions of practical device operation, should be adequately addressed. From what experimental feasibility is concerned with, the RCSJ model is usually considered to give a sensible description for the classical junction regime, $(E_C/E_J) \ll 1$ (see [12]), while the underdamped regime of the dynamics should be fine for low values of $\tilde{\epsilon}_\delta$. Both conditions can be easily fulfilled in tunnel junctions (see e.g. [25]). The parameters shown in the figures would correspond, assuming e.g. $R = 10\,\mathrm{k}\Omega$ and $I_c = 0.03\,\mu\mathrm{A}$, to $(E_C/E_J) \approx 10^{-3}$ and $\omega \approx 9\,\mathrm{GHz}$, just to give an idea about feasibility matters. As we have already mentioned, large regions of parameter space exhibit stable discrete breathers. Rotor localised modes, once adequately formed in the system, should be directly observable, by measuring mean voltage accross vertical links. Oscillator localised modes cannot be observed in that way, but their presence could perhaps be inferred by indirect effects.

Our results point toward the utmost importance of experimental work on Josephson-junction ladders, as a benchmark system for nonlinear concepts, in the firm belief that the interaction between theory and experiment is the correct way toward serious advances in Nonlinear Condensed Matter Physics.

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