Ratchet potential for fluxons in Josephson-junction arrays

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Abstract. – We propose a simple configuration of a one-dimensional parallel array of Josephson junctions in which the pinning potential for trapped fluxons lacks inversion symmetry (ratchet potential). This system can be modelised by a set of non-linear pendula with alternating lenghts and asymmetric harmonic couplings. We show, by molecular dynamics simulation, that fluxons behave as single particles in which the predictions for overdamped thermal ratchets can be easily verified.

Directional motion (DM) of Brownian particles has been a target of basic and applied research in the last five years. The initial motivation and interest in this field came from cell biology: The study of the mechanism of vesicles transport inside eukariotic cells, via motor proteins along microtubules [1, 2]. Later on new systems with the same underlying ideas for transportation were proposed. Those systems include: phase separation engines [3], drop motion under ac forces [4], growth of surfaces [5], and rectification in asymmetric superconducting rings (SQUID) [6]. To have DM of a particle submitted to a periodic one-dimensional potential $V(x) = V(x + L)$, one needs i) broken spatial symmetry, *i.e.* $V(x) \neq V(-x)$ and ii) to drive the particle out of thermal equilibrium. This last condition makes it possible to extract work from the system.

The first condition is usually fulfilled by the use of an *ad hoc* ratchet potential [2]. The latter condition can be implemented in different ways: by an oscillating driving field [7], by a time-correlated non-thermal noise [2] or a fluctuating $V(x)$ potential [8], among others. In all these cases, the conditions for the fluctuation-dissipation theorem do not hold.

In this paper we propose a *new experimental realization of DM in a very simple and controllable system*: a parallel Josephson-junction array. Such system has become an excellent realization of the Frenkel-Kontorova (FK) or discrete sine-Gordon model [9]. Fluxons (or
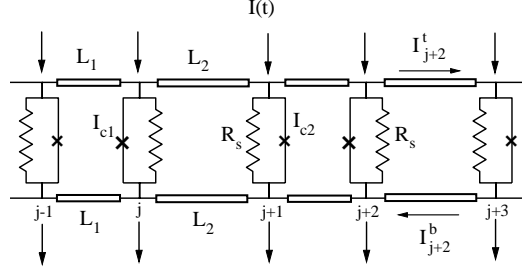
Fig. 1. – Schematic equivalent circuit of the parallel array of Josephson junction. External current is injected from top to bottom. R_s is a small shunt resistance to deal with overdamped junctions.

kinks) in the array move along a periodic pinning potential (the so-called Peierls-Nabarro potential). We will show that in a simple geometry this potential is a ratchet one and a fluxon behaves as a particle with well-defined directional motion.

   This letter is organized as follows: we begin by describing the basic assumptions for the array and the equations of motion for the phases are obtained. Then, we calculate the energy profile for a fluxon trapped in the array and show its "ratchet" character. Finally, we compare our results for the dynamics of the fluxon with what is expected from single-particle dynamics.

   We shall study first the array configuration. Let us consider a parallel array of Josephson-junctions with alternating critical currents (I_{c1} and I_{c2}, generally $I_{c\alpha}$) and alternating area plaquettes. The area differences cause alternating self-inductions L_1 and L_2. In order to deal with overdamped junctions, we shunt each of them with a small external resistance R_s so that we can neglect capacitance effects. Figure 1 is a schematic picture of the circuit. We denote by $2\pi x_j$ the phase difference across the junction j. $I(t)$ is an external driving current. The equation for current conservation through the junction j read as

$$\frac{V_j}{R_\mathrm{s}} + I_{c\alpha}\sin 2\pi x_j = I(t) + I_{j-1}^t - I_j^t = I(t) + I_{j-1}^b - I_j^b, \tag{1}$$

where

$$V_j = \Phi_0 \dot{x}_j, \tag{2}$$

where Φ_0 is the flux quantum. The phase diference along a plaquette is given by

$$x_{j+1} - x_j = \frac{\Phi_j}{\Phi_0}, \tag{3}$$

where

$$\Phi_j = \Phi^{\mathrm{ext}} - L_\alpha(I_j^t + I_j^b) \tag{4}$$

is the magnetic flux through a plaquette, sum of two contributions: external and induced magnetic fields [9]. We set now $\Phi^{\mathrm{ext}} = 0$. Combining eqs. (1), (2), (3) and (4), and dividing the array into two sublattices, the equations of motion for the phases are

$$\frac{\Phi_0}{R_\mathrm{s}}\dot{x}_j + I_{c1}\sin 2\pi x_j = I(t) + \frac{\Phi_0}{2L_1}(x_{j-1} - x_j) + \frac{\Phi_0}{2L_2}(x_{j+1} - x_j), \tag{5}$$

$$\frac{\Phi_0}{R_\mathrm{s}}\dot{x}_{j+1} + I_{c2}\sin 2\pi x_{j+1} = I(t) + \frac{\Phi_0}{2L_2}(x_j - x_{j+1}) + \frac{\Phi_0}{2L_1}(x_{j+2} - x_{j+1}). \tag{6}$$

Normalizing the equations by $I_L = \Phi_0/2L_1$ and putting $K = 4\pi I_{c1}L_1/\Phi_0$, $\alpha = I_{c2}/I_{c1}$ and $\beta = L_1/L_2$, we obtain the dimensionless equations of motion

$$\dot{x}_j + \frac{K}{2\pi}\sin 2\pi x_j = \tilde{I}(t) + (x_{j-1} - x_j) + \beta(x_{j+1} - x_j), \tag{7}$$

$$\dot{x}_{j+1} + \alpha\frac{K}{2\pi}\sin 2\pi x_{j+1} = \tilde{I}(t) + \beta(x_j - x_{j+1}) + (x_{j+2} - x_{j+1}), \tag{8}$$

where time derivatives are given in the time scale $\tau = 2L_1/R_s$. Here K play the role of discreteness parameter [9] and α and β measure the asymmetry of the array. Flux trapped in the array determines the boundary conditions, therefore, if we have n flux quanta in N plaquettes, then $x_{N+1} = x_1 + n$. Note that these equations of motion (7) and (8) correspond to the well-known FK model in a generalised asymmetric form. The overdamped dynamics for the symmetric model have been reviewed in ref. [10]. The mechanical analogue of the JJ array we are dealing with is a set of pendula with two different lengths coupled by two kinds of harmonic springs.

It is expected for $K \to 0$ that the pinning potential for fluxons vanishes. But we are interested in the opposite limit in which fluxons are strongly pinned to the discrete lattice. Due to the spatial asymmetry of the array we expect to have an asymmetric pinning potential.

Now we will try to describe the fluxon as a single particle in the pinning periodic potential. Fluxon location is given by its center of mass (CM) which can be expressed as [11]

$$X_{CM}(x_1, \ldots, x_N) = \frac{1}{2} + \sum_{i=1}^{N} i(x_{i+1} - x_i) \tag{9}$$

for any given phase configuration $\{x_i\}$. On the other hand, the energy of the array is the sum of two contributions: Josephson and magnetic energy. In the adimensional parameters defined above this energy reads as

$$E(x_1, \ldots x_N) =$$
$$\sum_{i=1,\text{odd}}^{N}\left[\frac{-K}{4\pi^2}(\cos 2\pi x_i + \alpha\cos 2\pi x_{i+1}) + \frac{\beta}{2}(x_{i+1} - x_i)^2 + \frac{1}{2}(x_{i+2} - x_{i+1})^2\right]. \tag{10}$$

From these expressions we can define $E(X_{CM}) = \min_{\{x_i\}} E(x_1 \ldots x_N)$ such that $\{x_i\}$ are kink configurations whose CM is X_{CM}.

In order to calculate this potential, several approaches have been tried in the context of non-linear discrete lattices. For low pinning (low K), a good estimation is obtained using the collective coordinates method [12]. Essentially, this method assumes a soliton profile for the $\{x_i\}$ corresponding to the continuous limit (sine-Gordon) and, in some cases, an analytical solution can be reached which only converges for low values of K. We have adopted a more general way to (numerically) obtain the potential profile $E(X_{CM})$. Maxima of this energy correspond to saddle-points in the N-dimensional phase space. For configurations containing one fluxon, $N-1$ directions are stable and one is unstable. Such maxima points can be obtained using standard minimax methods. We have performed a linear stability analysis around these points to get the direction of destabilization of the maximum energy configuration. Using this saddle configuration as initial condition, we perturb it along the unstable direction and study the relaxation (by numerical integration of eqs. (7) and (8), with $\tilde{I}(t) = 0$ to the adjacent minima. Then, the energy and CM along the trajectory are computed (according to eqs. (9) and (10)). Using this method we have been able to calculate the energy profile $E(X_{CM})$ for a

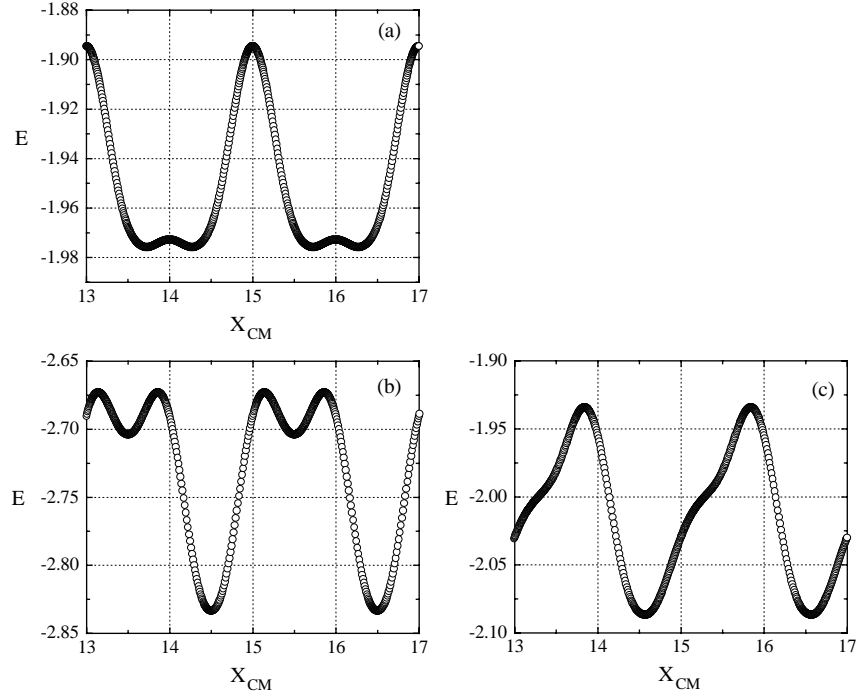Fig. 2. – Energy profile $E(X_{\mathrm{CM}})$ for different values of asymmetry parameters (α, β). (a) $\alpha = 0.5, \beta = 1.0$. Same plaquette areas and different critical currents. (b) $\alpha = 1, \beta = 0.5$. Different plaquettes and the same critical currents. (c) $\alpha = 0.5, \beta = 0.5$ gives "ratchet-like" potential.

trapped fluxon as a function of its CM. Figure 2 shows potential profiles for a set of the model parameters (K, α, β). In the following we set $K = 4.0$. For $\alpha = 0.5$ and $\beta = 1.0$ (different critical currents), the energy profile shows a double-well structure symmetric with respect to the bottom of the wells. The $\alpha = 1.0$, $\beta = 0.5$ (different areas) potential is symmetric with respect to the tops. As expected, for any other values the potential profiles do not show inversion symmetry. The values $\alpha = 0.5$ and $\beta = 0.5$ give a good approximation to the asymmetric sawtooth potential used in the literature [2]. It is interesting to note that, unlike other works in extented systems [11, 13], neither the on-site potential nor the interparticle potential are asymmetric in the field variables.

We will study now the dynamical behavior of a fluxon in the asymmetric lattice. For all simulations, we take $N = 30, n = 1, K = 4.0, \alpha = 0.5$ and $\beta = 0.5$. We drive the system out of thermal equilibrium by applying an external ac bias current. The $\tilde{I}(t)$ term in eqs. (7) and (8) is then expressed as

$$\tilde{I}(t) = \tilde{I}_{\mathrm{ac}} \sin \omega t + \xi(t). \tag{11}$$

Here, $\xi(t)$ is the white noise ($\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t')$)[1] which, in the absence of other forces, brings the system to thermal equilibrium. Thus, the equations of motion take the form of a system of stochastic differential equations. We have solved them using a fourth-order Runge-Kutta method for the deterministic part and a third-order one for the

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[1] Where $D = \frac{k_{\mathrm{B}}T}{\Phi_0^2/2L_1}$ is the temperature in units of the magnetic energy.
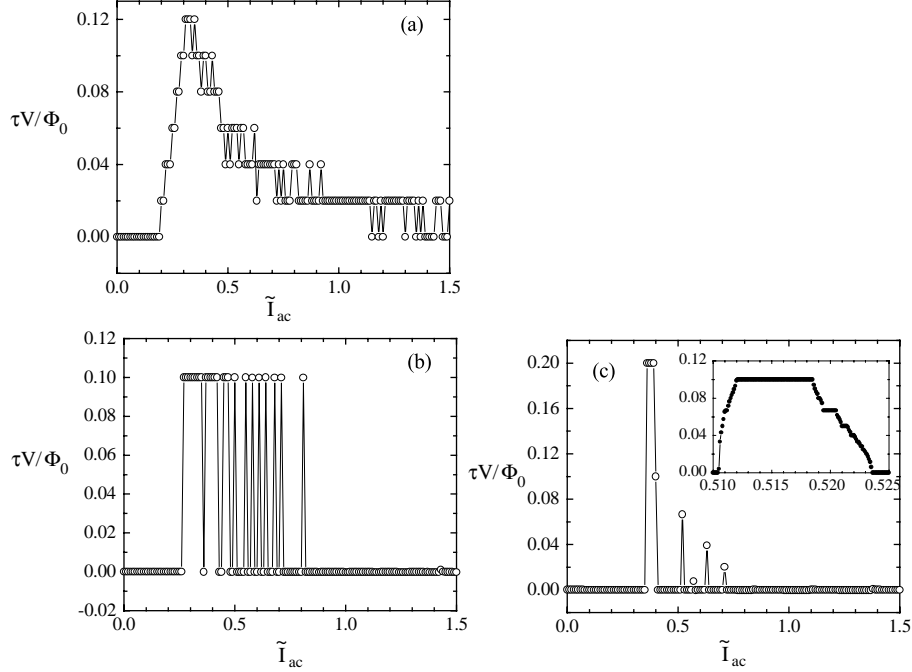
Fig. 3. – Voltage-current (IV) curves at $T = 0$ and ac current, for different frequencies ω in units of $1/\tau$. (a) $\omega = 2\pi \cdot 0.01$. (b) $\omega = 2\pi \cdot 0.05$. (c) $\omega = 2\pi \cdot 0.1$.

stochastic one [14].

We will concentrate first on the deterministic ($T = 0$) dynamics. We check the ratchet behavior measuring the critical current of the array under dc driving. For positive driving we find a fluxon depinning current $I_{\mathrm{dp}}^+ \approx 0.18$, whereas for negative $I_{\mathrm{dp}}^- \approx 0.31$ (for the symmetric case, $\alpha = 1.0$ and $\beta = 1.0$, $I_{\mathrm{dp}} \approx 0.19$). Under ac currents we should observe a IV curve showing rectification of the external current and the maximum efficiently for rectification (in the low-frequency limit) of the ac currents is expected to be around I_{dp}^+ [3]. Figure 3 shows the IV (voltage *vs.* amplitude I_{ac}) curves for $I_{\mathrm{dc}} = 0$ and $T = 0$ for different frequencies ω. Low-frequency response clearly resembles that found for a single particle [15]. As was noted in one-particle simulations, voltage is quantised:

$$V = \frac{\Phi_0}{\tau} \sum_{i=1}^{N} \langle \dot{x}_i \rangle = \frac{2\Phi_0}{\tau} \left(\frac{p}{q} \omega \right), \tag{12}$$

p and q being integer numbers [16]. For finer resolution, the IV curves appears to have a devil's staircase structure [15] (see inset of fig. 3c). This quantisation result can be straightforwardly obtained by assuming that, in the overdamped ac dynamics, the phases $x_i(t)$ can be expressed by a two-dimensional envelope function [10, 17] $x_i(t) = \zeta(i - Vt/\Phi_0, \omega t)$ [18]. Now, imposing energy balance along the trajectory

$$\sum_{i=1}^{N} \int_0^\infty \dot{x}_i^2 \mathrm{d}t = \sum_{i=1}^{N} \int_0^\infty \dot{x}_i \tilde{I}(t) \mathrm{d}t, \tag{13}$$

we find that only voltages commensurate with the current frequency give non-zero contribution
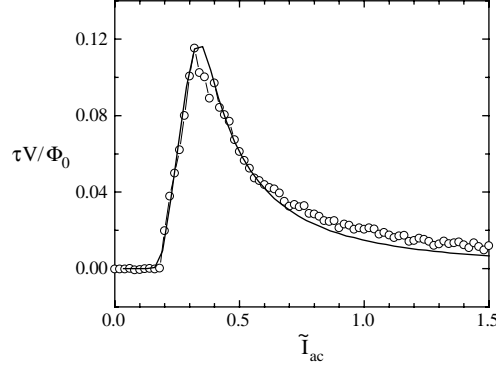
Fig. 4. – Voltage-current (IV) curves for ac current at $D = 0.001$ and $\omega = 2\pi \cdot 0.01$. The thick line shows the comparison with eq. (3) of ref. [2].

to the right side of eq. (13).

Quantised steps appear rounded at non-zero temperature as a consequence of multiple hops between dynamically close attractors, corresponding to the mode-locking trajectories [19]. In the adiabatic (low-frequency) and low-temperature limit we have fitted the IV curves with the solution of Magnasco [2] for a sawtooth ratchet potential. We have found a good fit (see fig. 4) for asymmetry ratio $\lambda_1/\lambda_2 = 0.56$ and energy barrier $\Delta E = 0.16$ obtained from the computed energy profile (see fig. 2c). At low frequencies the steps disappear, even for very low temperatures. At high frequency the steps seem to be more stable against thermal fluctuations and probably could be observed in real experiments.

In summary, we have proposed a new experiment for DM based in the motion of a fluxon in a one-dimensional Josephson-junction array. We have calculated the effective pinning potential for the fluxon and found the parameters needed to have almost a sawtooth one. The simulated IV curves mimic the behavior of a single particle in an asymmetric periodic potential.

The actual feasibility to fabricate these kinds of arrays along with the simplicity of our design make this experiment really accesible [20]. Even more, the possibility of introducing in the array a controlled number of fluxons could serve to study the influence of interaction in directional motion Brownian particles [21]. Preliminary results in this direction show that complex behavior dominates the multifluxon dynamics.

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