Taming Chaotic Solitons in Frenkel-Kontorova Chains by Weak Periodic Excitations

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(Received 29 July 2004; published 1 December 2004)

We have proposed theoretically and confirmed numerically the possibility of controlling chaotic solitons in damped, driven Frenkel-Kontorova chains subjected to additive bounded noise by weak periodic excitations. Theoretically, we obtained an effective equation of motion governing the dynamics of the soliton center of mass for which we deduced Melnikov's method-based predictions concerning the regions in the control parameter space where homoclinic bifurcations are frustrated. Numerically, we found that such theoretical predictions can be reliably applied to the original Frenkel-Kontorova chains, even for the case of *localized* application of the soliton-taming excitations, and there is strikingly good agreement between analytical estimates and numerical results.

DOI: 10.1103/PhysRevLett.93.237006 PACS numbers: 74.40.+k, 05.45.Gg

Control of chaos in spatially extended systems represents one of the most interesting and challenging problems in the field of nonlinear dynamics. Examples of possible applications include the stabilization of superconducting Josephson-junction arrays [1], semiconductor laser arrays [2], and periodic patterns in optical turbulence [3], to quote just a few. A fundamental question in this field is whether effective control of the whole system can be achieved by solely influencing a part of its freedom degrees. In this regard, *localized* control of spatiotemporal chaos has only recently been investigated in diverse extended systems, such as plasmas [4], coupled oscillators [5,6], isotropic systems [7], continuous extended systems [8], and coupled map lattices [9]. Much less effort has been devoted to control chaotic *localized* solutions (such as solitons and breathers) arising from nonlinear partial differential equations [10–12]. Although diverse works concerning similar chaotic localized solutions in *lattices* of nonlinear oscillators exist [13], to the best of our knowledge their control (in the sense of regularization), which is a problem of general interest [14], has not been considered before.

In this Letter, we present a general theoretical approach to control chaotic solitons in damped, noisy, and driven Frenkel-Kontorova (FK) chains by weak periodic excitations. It is worth mentioning that the FK model provides a fairly accurate description of diverse physical and biological systems and phenomena, including charge density wave conductors, ladder networks of discrete Josephson junctions, and DNA dynamics, to quote just a few [15]. For the sake of concreteness, the approach is discussed, in particular, in the simple case of linear damping, additive noise, and two harmonic forces

$$\ddot{u}_j = -\frac{K}{2\pi}\sin(2\pi u_j) + u_{j+1} - 2u_j + u_{j-1} - \alpha\dot{u}_j + F\cos(\omega t) + \beta F\cos(\Omega t + \varphi) + \xi(t), \quad (1)$$

where u_j is the phase of the jth oscillator, α is the damping coefficient, K measures the strength of the substrate potential, $F\cos(\omega t)$ is the chaos-inducing excitation, and (β, Ω, φ) are the as yet undetermined suitable parameters of the chaos-taming excitation $\beta F\cos(\Omega t + \varphi)$. Here, $\xi(t) \equiv \lambda\sin[\Omega' t + \sigma B(t) + \Gamma]$ is a bounded noise term with zero mean, where λ and Ω' are the amplitude and averaged frequency, respectively, $B(t)$ is a unit Wiener process, σ represents the intensity of random frequency, and Γ is a random initial phase uniformly distributed in $[0, 2\pi)$. Also, a finite chain of N particles with the following boundary condition: $u_{j+N} = u_j + N + 1$ is assumed to keep the analysis close to experimental realization (e.g., a circular array of Josephson junctions). As is well known, a collective coordinate formalism (CCF) [16,17] permits one to describe the motion of the soliton center of mass, $X(t)$, by means of an *effective* ode, which is a perturbed pendulum for the FK model [18]. Thus, the application of CCF to Eq. (1) by assuming a sine-Gordon profile for the (discrete) soliton, $u_n = n \pm (2/\pi)\tan^{-1}\{\exp[n - X(t)]/l_0\}$, yields the perturbed pendulum equation

$$\frac{d^2z}{d\tau^2} = -\sin z - \delta\frac{dz}{d\tau} - \gamma\cos(\tilde{\omega}\tau) - \beta\gamma\cos(\tilde{\Omega}\tau + \varphi) - \tilde{\lambda}\sin[\tilde{\Omega}'\tau + \tilde{\sigma}B(\tau) + \Gamma] \quad (2)$$

where $z \equiv 2\pi X$, $\tau \equiv \Omega_{PN}t$, $\delta \equiv \alpha\Omega_{PN}^{-1}$, $\gamma \equiv \pi^3 l_0 F\Omega_{PN}^{-2}$, $\tilde{\omega} \equiv \omega\Omega_{PN}^{-1}$, $\tilde{\Omega} \equiv \Omega\Omega_{PN}^{-1}$, $\tilde{\lambda} \equiv \pi^3 l_0\lambda\Omega_{PN}^{-2}$, $\tilde{\Omega}' \equiv \Omega'\Omega_{PN}^{-1}$, and $\tilde{\sigma} \equiv \sigma\Omega_{PN}^{-1/2}$, where Ω_{PN} and l_0 are the Peierls-Nabarro frequency and the soliton width, respectively $[\Omega_{PN}^2/2\pi = (\pi^3 + 2\pi^5 l_0^2)/(6l_0^4\sinh(\pi^2 l_0))]$. Let us assume in the following that the forcing, noise, and damping terms in Eq. (2) are small amplitude perturbations of the underlying integrable pendulum, i.e., they satisfy Melnikov's method (MM) requirements [19], and that, in the absence of both chaos-taming excitations ($\beta = 0$) and noise ($\lambda = 0$), the perturbed pendulum exhibits homoclinic chaos which corresponds to a chaotic soliton existing in the FK model (1). Now we let the chaos-taming excitation act on each pendulum of the FK chain [and hence on the associated perturbed pendulum (2)] ($\beta > 0$) in the presence of noise ($\lambda > 0$) and deduce analytical estimates of the regularization regions in the parameter space ($\beta, \tilde{\Omega}, \varphi$). Next, we conjecture, notwithstanding the *approximate and perturbative character* of both CCF and MM, that such theoretical predictions could remain reliable to *some* extent [in the corresponding regions of the control parameter space (β, Ω, φ)] to effectively tame chaotic solitons arising from the uncontrolled FK model (1). As shown below, this is just the case. For the sake of clarity, we shall consider separately the cases with and without noise.

Purely deterministic case.—In the absence of noise ($\lambda = \tilde{\lambda} = 0$), the application of MM to Eq. (2) gives us the Melnikov function (MF)

$$M^{\pm}_{\tilde{\lambda}=0}(\tau_0) = -C \mp A\cos(\tilde{\omega}\tau_0) \mp B\cos(\tilde{\Omega}\tau_0 + \varphi) \quad (3)$$

where $C \equiv 8\delta$, $A \equiv 2\pi\gamma \mathrm{sech}(\pi\tilde{\omega}/2)$, and $B \equiv 2\pi\beta\gamma\mathrm{sech}(\pi\tilde{\Omega}/2)$, where the plus (minus) sign corresponds to the upper (lower) homoclinic orbit of the unperturbed pendulum. Now the previous theory on chaos suppression by weak periodic perturbations [5,25] directly applies to the above MF. In particular, for the main resonance case $\Omega/\omega \equiv \tilde{\Omega}/\tilde{\omega} = 1$, one obtains the *boundary function* in the $\varphi - \beta$ parameter plane,

$$\beta = -\cos\varphi \pm \sqrt{\cos^2\varphi - (1 - C^2/A^2)} \quad (4)$$

where homoclinic chaos is suppressed. Note that Eq. (4) represents a sufficient condition for $M^{\pm}_{\tilde{\lambda}=0}(\tau_0)$ to be negative or null for all τ_0, and that the area enclosed by such a boundary is $4|C|/A$. Next, one can compare the theoretical predictions and Lyapunov exponent (LE) calculations of both the FK model and the associated perturbed pendulum. It is worth noting that one cannot expect too good a quantitative agreement between the two kinds of results because LE provides information concerning solely steady chaos, while MM is a perturbative method generally related to transient chaos. The maximal LE, Λ^+, was calculated for each point on a dense grid, with (normalized) initial phase φ and amplitude β along the horizontal and vertical axes, respectively, for both systems. An illustrative example is shown in Fig. 1. Typically, one finds for *both* systems that complete regularization $[\Lambda^+(\beta > 0) \leqslant 0]$ mainly appear inside maximal islands which *symmetrically* contain the theoretically predicted areas where even the chaotic transients are suppressed. Note the great similarity of both LE distributions in their respective $\varphi - \beta$ parameter planes.

Effect of additive bounded noise.—In the presence of noise ($\lambda, \tilde{\lambda} > 0$), one has a random Melnikov process (RMP) [22–24]:

$$M^{\pm}_{\tilde{\lambda}>0}(\tau_0) = M^{\pm}_{\tilde{\lambda}=0}(\tau_0) \mp R(\tau_0) \quad (5)$$

where $R(\tau_0) \equiv 2\int_{-\infty}^{\infty} \mathrm{sech}(\tau)\xi(\tau + \tau_0)d\tau$ is the component of the RMP due to noise. It is well known that the simple zeros of a (deterministic) MF imply transversal intersections of stable and unstable manifolds, giving rise to Smale horseshoes and hence hyperbolic invariant sets

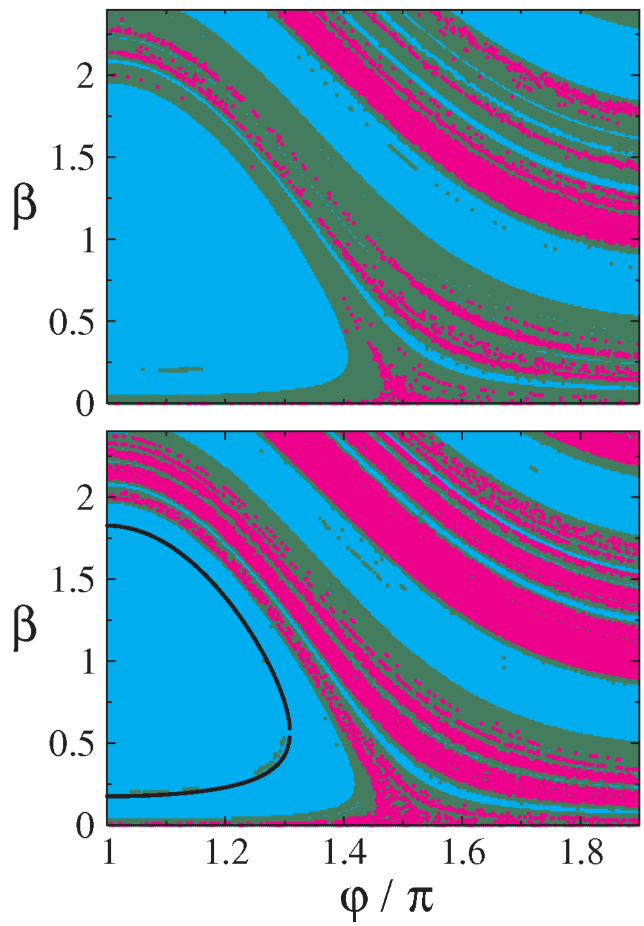

FIG. 1 (color online). Maximal LE distribution in the ($\varphi - \beta$) parameter plane for the soliton in the FK model (Eq. (1), top panel) and for the associated perturbed pendulum (Eq. (2), bottom panel) in the absence of noise ($\lambda = 0$). Cyan (light gray), green (dark gray), and magenta (halftone gray) regions indicate that the respective maximal LE, $\Lambda^+(\beta > 0)$ is nonpositive, belongs to the interval $[0, \Lambda^+(\beta = 0)]$, and is larger than $\Lambda^+(\beta = 0) \simeq 0.03$ bits/s, respectively. The solid black contour indicates the predicted boundary function [cf., Eq. (4)]. Fixed parameters: $K = 1$, $\alpha = 0.1$, $F = 0.0017$, $\omega = 0.04\pi$, and $l_0 = 1.0118179$. Only LEs corresponding to values $\varphi \in [\pi, 2\pi]$ are depicted because of symmetry with respect to the optimal suppressory value $\varphi_{\mathrm{opt}} \equiv \pi$.

[20,21]. However, the RMP in Eq. (5) can be considered only in some statistical sense. Thus, one has to deduce an *effective* MF (and hence to discuss the existence of its simple zeros) to obtain *approximate* predictions concerning the RMP. To this end, we assume that in the absence of periodic excitations ($\gamma = 0$), the threshold amplitude of bounded noise excitation for the onset of chaos occurs when the RMP has a simple zero in the mean-square sense [24], i.e.,

$$C^2 = \sigma_R^2, \qquad \sigma_R^2 \equiv 8\pi^2 \int_0^\infty S(\hat{\omega})\mathrm{sech}^2\left(\frac{\pi\hat{\omega}}{2}\right) d\hat{\omega} \quad (6)$$

where:

$$S(\hat{\omega}) \equiv \frac{\tilde{\lambda}^2\tilde{\sigma}^2}{4\pi} \frac{\hat{\omega}^2 + \tilde{\Omega}'^2 + \tilde{\sigma}^4/4}{\left(\hat{\omega}^2 - \tilde{\Omega}'^2 - \tilde{\sigma}^4/4\right)^2 + \tilde{\sigma}^4\hat{\omega}^2} \quad (7)$$

is the spectral density of $\xi(t)$. We found that this estimate is reasonably confirmed by numerical simulations. Now, in the presence of periodic excitations ($F, \beta > 0$), we define an effective MF

$$M_{\mathrm{eff}}^{\pm}(\tau_0) \equiv -C_{\mathrm{eff}} \mp A\cos(\tilde{\omega}\tau_0) \mp B\cos\left(\tilde{\Omega}\tau_0 + \varphi\right) \quad (8)$$

with

$$C_{\mathrm{eff}} \equiv C - \sigma_R \quad (9)$$

so that

$$M_{\tilde{\lambda}>0}^{\pm}(\tau_0) \leq M_{\mathrm{eff}}^{\pm}(\tau_0), \ \forall \ \tau_0. \quad (10)$$

It is worth noting that Eq. (10) connects the effective MF with the RMP. Thus, one can again apply the previous theory [5,25] to the above effective MF [Eq. (8)]. In particular, for the main resonance case discussed above for a purely deterministic situation, one obtains a new boundary function

$$\beta = -\cos\varphi \pm \sqrt{\cos^2\varphi - (1 - C_{\mathrm{eff}}^2/A^2)}, \quad (11)$$

which represents a sufficient condition for $M_{\mathrm{eff}}^{\pm}(\tau_0) \leq 0, \forall\tau_0$, and *hence* for $M_{\tilde{\lambda}>0}^{\pm}(\tau_0) \leq 0, \forall\tau_0$ [cf., Equation (10)]. Thus, a first prediction is that the theoretical boundaries of the regularization regions associated with the random and deterministic cases have *identical* form and are symmetric with respect to the *same* (single) optimal suppressory value $\varphi_{\mathrm{opt}} \equiv \pi$, while the respective enclosed areas are *lesser* for the former case than for the latter case [cf., Eqs. (4) and (11)]. A second prediction is that there exists a critical amount of noise, $[\sigma_R]_c \equiv C$, from which regularization is not possible any more, and that this critical value depends upon the damping strength, as expected. Figure 2 shows an illustrative example of the comparison between theoretical predictions and LE calculations. As for the deterministic limiting case, one typically obtains an extraordinary agreement between the two kinds of results.

Additionally, we investigated the robustness of the above theoretical predictions against a *discrete* (not a global one) application of the chaos-taming excitations.

Since FK solitons present a very sharp spatial localization (typically, $l_0 \sim 1$ in our simulations), one could expect that a reliable soliton control may be achieved by solely applying the soliton-taming excitation to a few pendula of the FK chain. To check this conjecture, we simulated rather large FK chains in parameter space regions where a chaotic soliton exists. Figure 3 shows an illustrative example of a chain of 200 pendula with soliton-taming excitations acting on every fiftieth pendulum in the absence of noise. We found that the regularization region in the $\beta - \varphi$ parameter plane has very approximately the same size as in the case of a global control [Compare top panels of Figs. 1 and 3]. To understand the mechanism underlying the regularization of the chaotic soliton, we calculated the temporal series of the soliton center of mass, $X(t)$, for a β constant while the control initial phase changed according to $\varphi(t) = N(t)/N_T$, where N_T and $N(t) \equiv \omega t/(2\pi)$ are the total number of driving cycles and the number of cycles after a time t, respectively. The bottom panel in Fig. 3 shows a representative example for $N_T = 200$. Starting from $\varphi = 0$, one sees that the soliton moves chaotically along the chain at φ values that are out of the predicted regularization region, as expected. For φ values belonging to the predicted regularization region, one typically observes

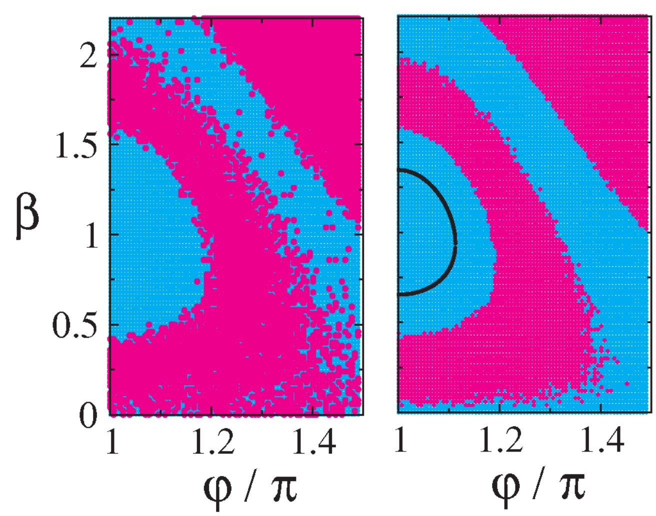

FIG. 2 (color online). Maximal LE distribution in the $\varphi - \beta$ parameter plane for the FK model (Eq. (1), left panel) and for the associated perturbed pendulum [Eq. (2), right panel] in the presence of noise ($\tilde{\lambda} = 1.5, \tilde{\Omega}' = 0.4\pi, \tilde{\sigma} = 0.2$). Cyan (light gray) and magenta (halftone gray) regions indicate that the respective maximal LE, $\Lambda^+(\beta > 0)$, is nonpositive and positive, respectively. The solid black contour indicates the predicted boundary function [cf., Equation (11)]. The remaining parameters are as in the caption of Fig. 1.

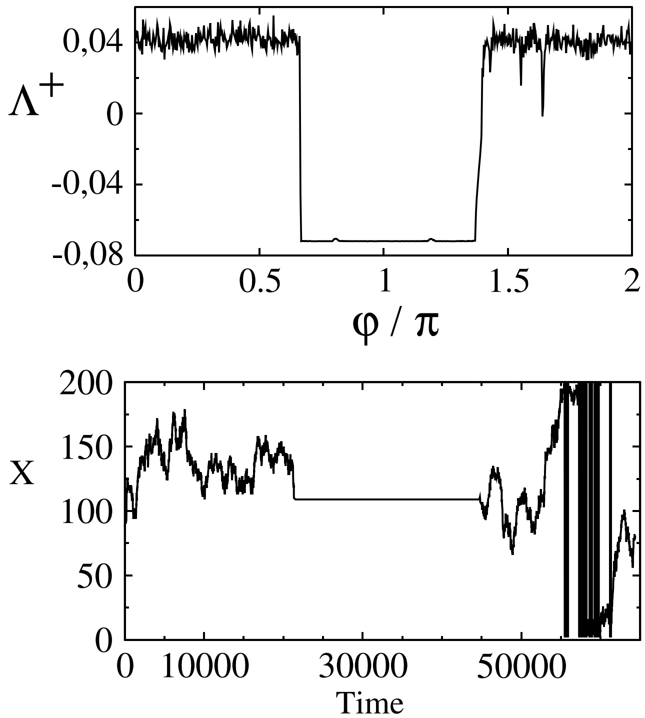

FIG. 3. Maximal LE for the FK model (Eq. (1), top panel) with 200 pendula and chaos-taming excitations acting on every fiftieth pendulum, and temporal series of the soliton center of mass (bottom panel), $X(t)$, while the initial phase varies according to $\varphi = \omega t/(400\pi)$ (see the text). $\lambda = 0$, $\beta = 0.6$, and the remaining parameters are as in the caption of Fig. 1.

that the soliton moves to be pinned to the *nearest* (regarding its position when crossing the chaotic threshold) pendula subjected to the soliton-taming excitation where it remains regularized. Finally, for certain φ values which are above the predicted regularization range, we found that the soliton moves with a definite (mode-locked) velocity along the chain while its behavior remains chaotic.

In summary, a general theoretical approach has been presented concerning the control of chaotic solitons in damped driven FK chains by weak periodic excitations in absence but also subjected to additive bounded noise. We have demonstrated the theory in a simple realistic situation, but in a general framework; quantitative details of cases not discussed here (e.g., general resonances $\Omega = p\omega/q$ and nonharmonic excitations), will be published elsewhere. We expect that the present results will stimulate experimental work, especially in circular arrays of Josephson junctions. Because of the generality of the present theoretical approach we also expect it can be applied to other types of lattices as well as to the cases of multiplicative noise and parametric chaos-taming excitation. Our current work is aimed at exploring theses cases.

P. J. M. and R. Ch. acknowledge financial support from the Spanish MCyT and European Regional Development Fund (FEDER) program through BFM2002-00113 and BFM2002-00010 projects, respectively.

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