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Chaos suppression and desynchronization phenomena in periodically coupled pendula subjected to localized heterogeneous forces

R. Chacón [a,*], P.J. Martínez [b], J.A. Martínez [c], S. Lenci [d]

[a] Departamento de Física Aplicada, Escuela de Ingenierías Industriales, Universidad de Extremadura, Apartado Postal 382, E-06071 Badajoz, Spain
[b] Departamento de Física Aplicada, E.U.I.T.I., Universidad de Zaragoza, Spain and Instituto de Ciencia de Materiales de Aragón, CSIC-Universidad de Zaragoza, E-50009 Zaragoza, Spain
[c] Departamento de Ingeniería Eléctrica, Electrónica y Automática, E.P.S., Universidad de Castilla-La Mancha, E-02071 Albacete, Spain
[d] Dipartimento di Architettura, Costruzioni e Strutture, Università Politecnica delle Marche, via Brecce Bianche, 60131 Ancona, Italy

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ABSTRACT

The suppressory effects of localized heterogeneous periodic pulses on the chaotic behaviour of *sinusoidally* coupled nonlinear pendula are studied. We assume that when the pendula are driven synchronously, i.e., all driving pulses have the same waveform, the chains display chaotic dynamics. It is shown that decreasing the impulse transmitted by the pulses of a minimal number of pendula results in regularization with the whole array exhibiting frequency synchronization over a wide range of coupling periods. These findings demonstrate that decreasing the impulse transmitted by localized external forces can tame chaos and lead to frequency-locked states in networks of *periodically coupled* dissipative systems.

1. Introduction

One of the pervasive notions of nonlinear science with a wide area of practical applications as well as fundamental theoretical issues is that of synchronization (and desynchronization) in networks of coupled oscillators [1–4]. Closely related with synchronization–desynchronization transitions in the context of chaotic arrays is the ubiquitous problem of controlling chaos [5]. In this regard, studies have shown that chaos in coupled arrays of damped, periodically forced, nonlinear oscillators can be tamed by disorder [6], impurities [7], localized controlling resonant forces [8,9], random shortcuts [10], and global disordered driving forces [11]. Generally, the arrays studied have been assumed to have homogeneous forces. In more realistic situations, however, the periodic forces acting on the oscillators often exhibit heterogeneous distributions having distinct periods, amplitudes, and wave forms. The heterogeneity-induced regularization of homogeneous chains of linearly coupled chaotic pendula by decreasing the impulse transmitted by the driving forces acting on a minimal number of oscillators has been recently demonstrated in [12]. It was also shown that, as the transmitted impulse is decreased, there typically emerges regular, frequency-locked dynamics, while desynchronization is due to the dispersion in the oscillator's amplitude.

2. Periodically coupled pendula

In the present work, we explore the stability of such a regularization scenario when the chaotic oscillators are *periodically* coupled. Diverse dynamic aspects of systems subjected to periodic coupling have been considered previously (see, e.g.,

* Corresponding author.
E-mail address: rchacon@unex.es (R. Chacón).

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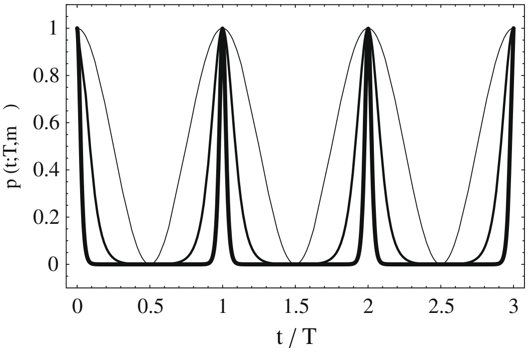

Fig. 1. Pulse function $p(t;T,m) \equiv \mathrm{cn}^2[2K(m)t/T;m]$ (cf. Eq. (1)) vs t/T for $m = 0$ (thin line), $m = 0.9993$ (medium line), and $m = 1 - 10^{-14}$ (thick line).

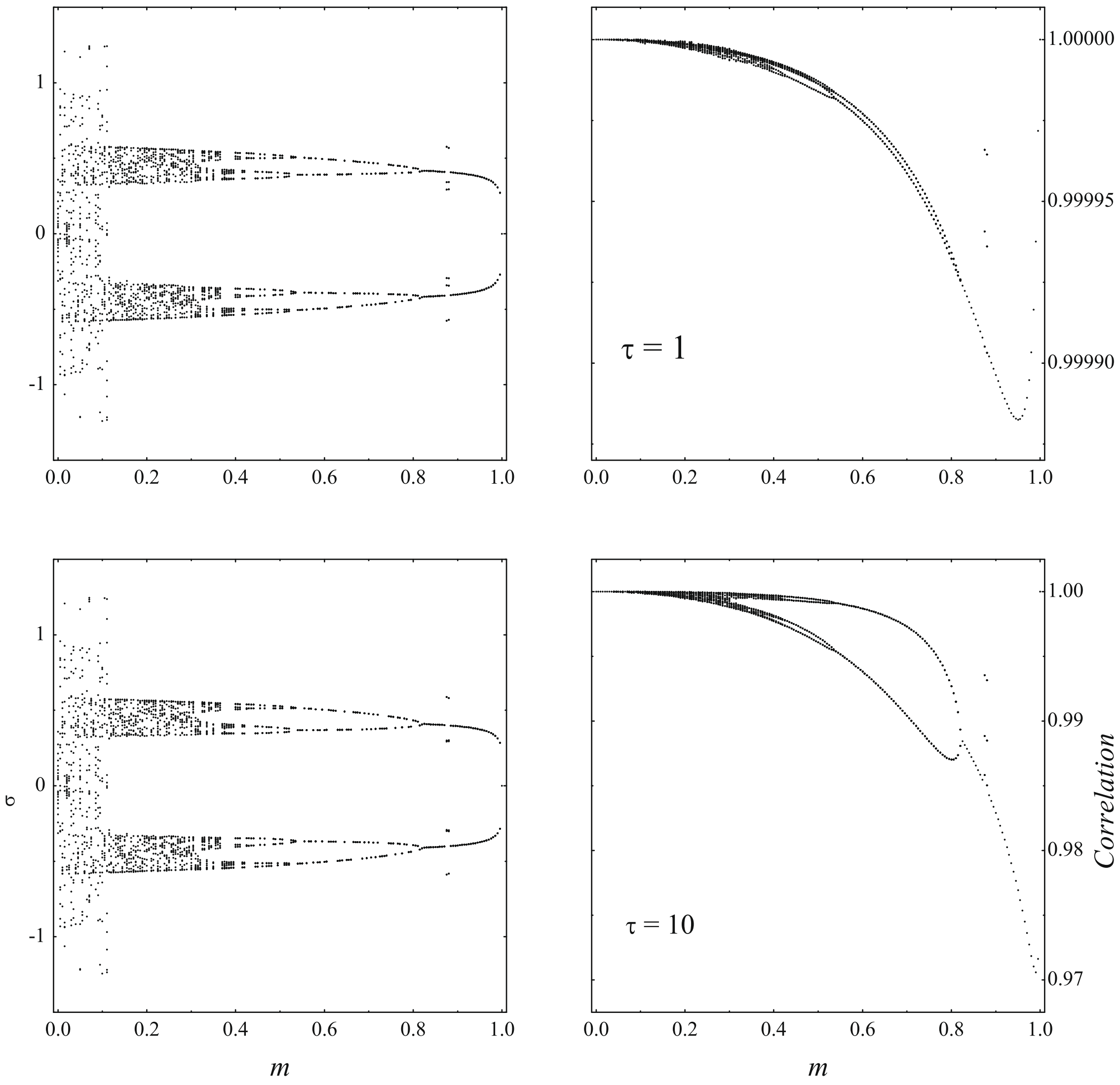

Fig. 2. Bifurcation diagram of the average velocity σ and correlation function as a function of the shape parameter m for a chain of $N = 5$ pendula, coupling $k_F = 3$, and two values of the coupling period.

[13,14]). For the sake of completeness, the results are discussed through the analysis of one-dimensional chains of damped kicked rotators. In particular, the chain is described by the dimensionless equation of motion

$$\ddot{\theta}_n + F\mathrm{cn}^2(\Omega t; m)\sin\theta_n = -\delta\dot{\theta}_n + k\{\sin[\Omega'(\theta_{n+1} - \theta_n)] + \sin[\Omega'(\theta_{n-1} - \theta_n)]\}, \tag{1}$$

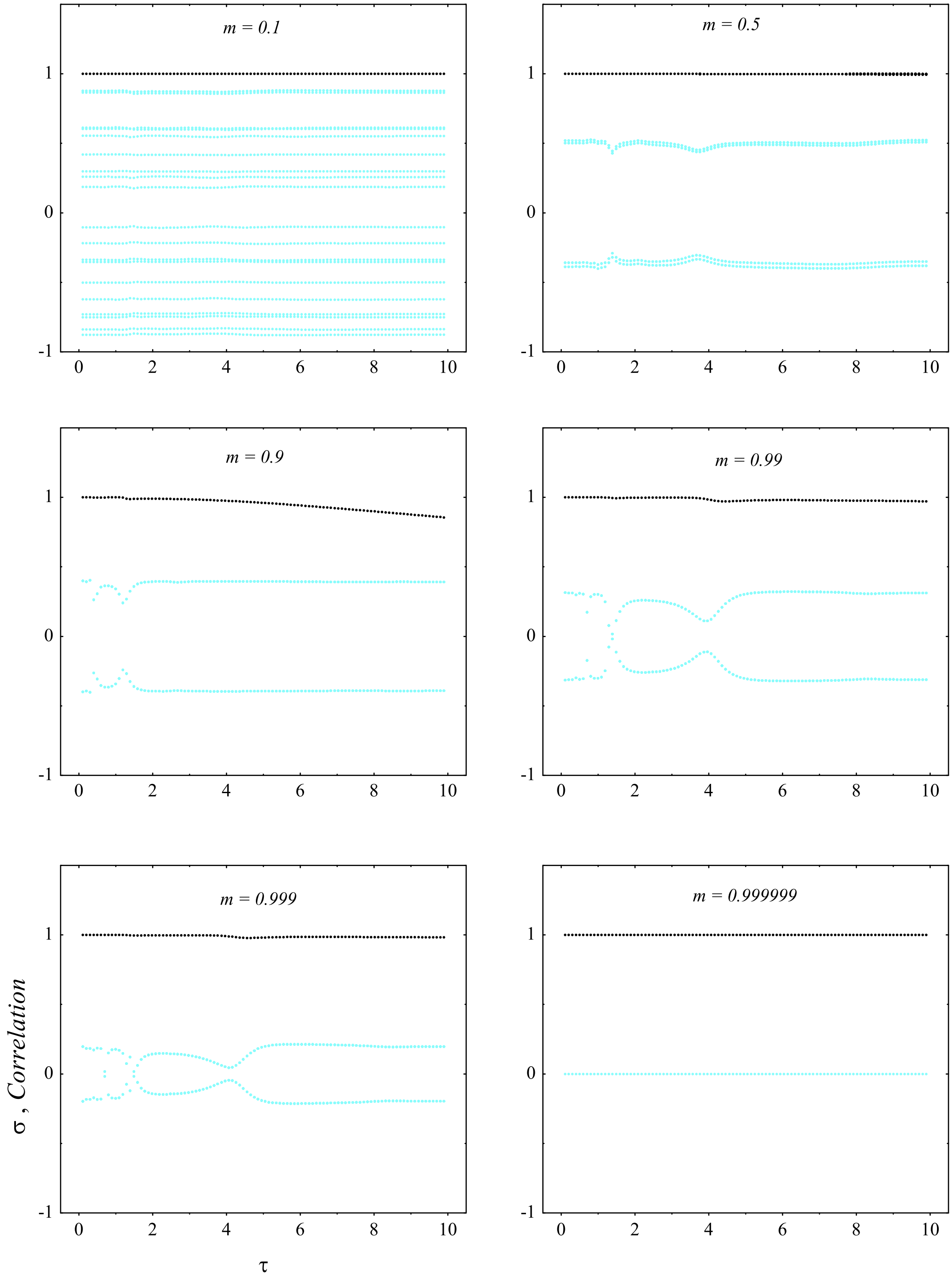

Fig. 3. Bifurcation diagram of the average velocity σ and the correlation function as a function of the coupling period τ for a chain of $N = 5$ pendula, coupling $k_F = 3$, and six values of the shape parameter.

where $n = 1, 2, \ldots, N, \Omega = \Omega(T, m) \equiv 2K(m)/T, T$ and F are the forcing period and amplitude, respectively, δ is the damping coefficient, k is the coupling constant, $\mathrm{cn}(\cdot; m)$ is the Jacobian elliptic function of parameter $m, K(m)$ is the complete elliptic integral of the first kind, $\Omega' \equiv 2\pi/\tau, \tau$ is the coupling period, and the shape parameter is taken to be $m = 0$ except for the two (free) end pendula which are subjected to pulses of variable width ($m \in [0, 1]$). The waveform of the pulse is varied solely by changing m between 0 and 1, such that by increasing m the pulse becomes ever narrower, until for $m \simeq 1$ one recovers a periodic sharply kicking excitation very close to the periodic δ-function, but with finite amplitude and width as in real-world impacts (see Fig. 1). Observe that $\mathrm{cn}^2(\Omega t; m = 0) = \cos^2(\pi t/T)$ while in the other limit, $m = 1$, the pulse area vanishes. Eq. (1) can be written in terms of the scaled dimensionless time $t' \equiv t\sqrt{F}$, which implies that there are *five* independent parameters

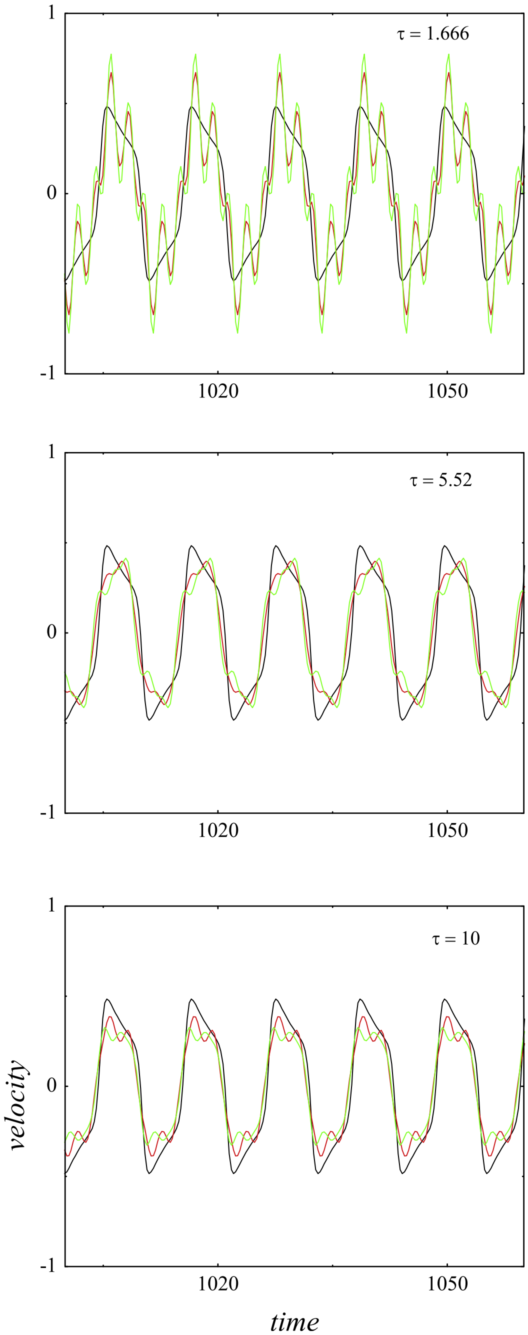

Fig. 4. Angular velocities of the end (black), next-to-end (red), and central (green) pendula as a function of time for a chain of $N = 5$ pendula, coupling $k_F = 3$, shape parameter $m = 0.999$, and three values of the coupling period: $\tau = 1.666, \tau = 5.52 \equiv T_F$, and $\tau = 10$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this paper.)

in this model: the coupling period τ, the number of rotators N, the scaled damped coefficient $\delta_F \equiv \delta/\sqrt{F}$, the scaled forcing period $T_F \equiv T\sqrt{F}$, and the scaled coupling constant $k_F \equiv k/F$. For the parameter values used in the present numerical simulations ($T_F = 5.52, \delta_F = 0.2$), each isolated pendulum driven by trigonometric pulses ($m = 0$) displays chaotic behaviour characterized by a positive Lyapunov exponent [15]. Eq. (1) was numerically integrated using a fourth-order Runge–Kutta algorithm with a time step $dt' = 0.001$. A useful measure that allows one to visualize the average global spatio-temporal dynamics of the chains is the average velocity

$$\sigma(jT_F) \equiv \frac{1}{N}\sum_{n=1}^{N}\frac{d\theta_n}{dt'}(jT_F), \tag{2}$$

where j is an integer multiple of the pulse period T_F, while the degree of synchronization is characterized by the correlation function

$$C(jT_F) \equiv \frac{2}{N(N-1)}\sum_{(il)}\cos\langle\theta_i(jT_F) - \theta_l(jT_F)\rangle, \tag{3}$$

where the summation is over all pairs of rotators. Notice that $C(t)$ is 1(0) for the perfectly synchronized (desynchronized) state.

Fig. 2 shows the average velocity as well as the correlation function at $t' = 970T_F, \ldots, 1000T_F$ versus the shape parameter for a chain of five pendula. Typically, the individual pendula go from chaos to stable equilibrium(oscillator death) as the shape parameter increases from 0 to 1 while the whole chain goes from perfect synchrony(at $m = 0$) to perfect trivial synchrony(at $m = 1$) passing through desynchronized states for $m \in]0, 1[$. The evolution of the desynchronized states is characterized by the correlation function undergoing an inverse period-doubling route as the shape parameter is increased, which corresponds to the scenario found in the case of linear coupling [12]. While the strength of desynchronization generally increases as the coupling period is increased, one typically finds that the correlation function exhibits a minimum as a function of the shape parameter (at $m_{\min}$) for sufficiently large coupling constant values (such as for $k_F = 3$, cf. Fig. 2). At this minimum, all the pendula present a $2T$ periodic solution while their amplitude distribution reaches a maximum range. Also, for certain values of the shape parameter (such as for $m = 0.99, 0.999$, cf. Fig. 3), the correlation function exhibits a weak minimum as a function of the coupling period at $\tau = \tau_{\min} \lesssim T_F$ (see Fig. 3). This maximum desynchronization is due to the dispersion in both the amplitude and waveform of the pendula response (see Fig. 4) and is a consequence of the competition between the two temporal scales of the problem (coupling and parametric excitation).

Similarly to the case of a linear coupling [12], the increase of the shape parameter has a twofold effect on the chaotic chains, which permits one to understand the appearance of a maximum desynchronization (at $m_{\min}$) as this parameter is varied, provided both the coupling constant and the coupling period are sufficiently large [hereafter referred to as strong coupling constant (SCC) and strong coupling period (SCP) regimes, respectively]. Indeed, while increasing the shape parameter from 0 improves the desynchronization-induced regularization of the chain, in the sense that to optimize the frequency-locking to the forcing, it simultaneously increases the heterogeneity-induced desynchronization of the chain by increasing the amplitude dispersion on the one hand, and the reshaping-induced oscillation death on the other. Indeed, the latter effect becomes dominant for sufficiently narrow pulses: the equilibrium $(\theta_n, \dot{\theta}_n) = (0, 0)$, unstable when the pendula are uncoupled, becomes attracting and suppresses the $2T$-periodic oscillations via an inverse supercritical Hopf bifurcation [15]. The dependence of the impulse transmitted at the pulse waveform corresponding to maximal desynchronization,

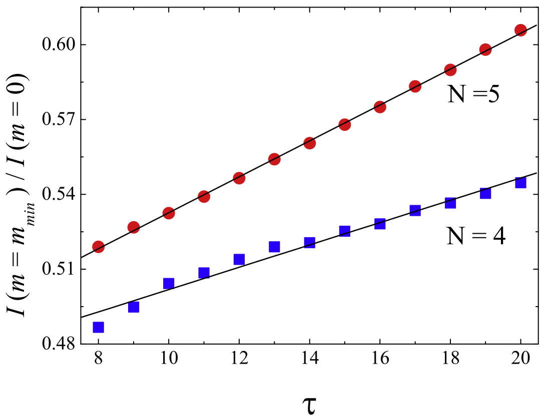

Fig. 5. Normalized transmitted impulse associated with maximal desynchronization $I(m = m_{\min})/I(m = 0)$ as a function of the coupling period τ for $N = 4$ (squares) and $N = 5$ (circles) pendula. Black lines denote linear $(0.45708 + 0.00447\tau, 0.4606 + 0.0072\tau)$ fits for $N = 4, 5$, respectively. Coupling $k_F = 3$.

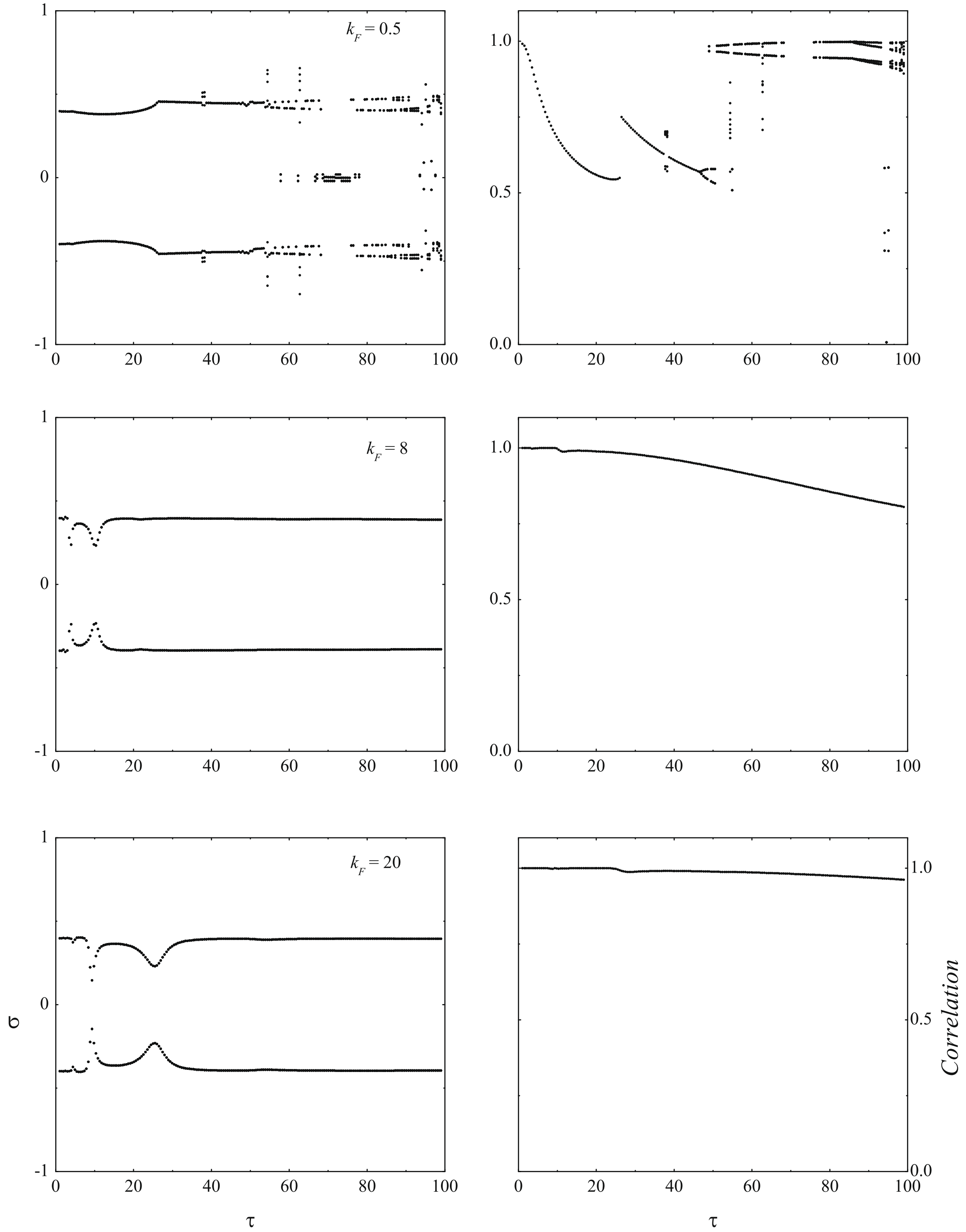

Fig. 6. Bifurcation diagram of the average velocity σ and the correlation function as a function of the coupling period τ for a chain of $N = 5$ pendula, shape parameter $m = 0.9$, and three values of the coupling constant.

$$I(m = m_{\min}) \equiv F \int_0^T \mathrm{cn}^2(\Omega t; m = m_{\min}) dt, \tag{4}$$

on the coupling period is shown in Fig. 5. We typically find a *linear* law in the SCP regime, which exhibits an increasing slope as the chain size is increased (see Fig. 5). For small coupling constants [hereafter referred to as a weak coupling constant

(WCC) regime], one observes the regularization of chaotic chains from small values of the coupling period to values belonging to the SCP regime, while a further increase of the coupling period leads to the pendula being chaotic again (such as for $k_F = 0.5$, cf. Fig. 6). This rather complex scenario disappears in the SCC regime, where desynchronization of the regularized states increases as the coupling period increases in the SCP regime (such as for $k_F = 8$, cf. Fig. 6). This desynchronization increase is less noticeable in the deep SCC regime (such as for $k_F = 20$, cf. Fig. 6) and is due to the dispersion in the pendula's amplitude, as can be appreciated in Fig. 7.

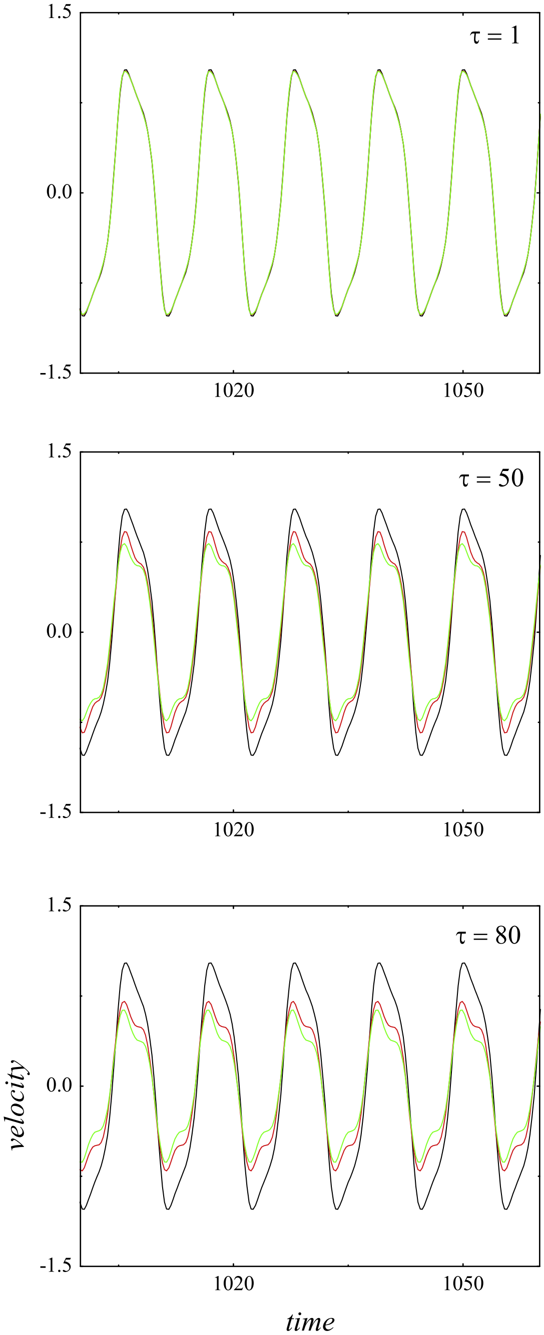

Fig. 7. Angular velocities of the end (black), next-to-end (red), and central (green) pendula as a function of time for a chain of $N = 5$ pendula, coupling constant $k_F = 8$, shape parameter $m = 0.9$, and three values of the coupling period: $\tau = 1$, $\tau = 50$, and $\tau = 80$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this paper.)

For small values of the coupling constant, the weak coupling period (WCP) regime exhibits a great complexity, as can be appreciated in the instance shown in Fig. 8. Indeed, one typically finds windows where the chains present chaotic desynchronized states interspersed with windows containing regular states with different degrees of synchronization (such as for $k_F = 0.2$, cf. Fig. 8) as well as abrupt chaos-order transitions via crisis phenomena (such as for $k_F = 0.5$, cf. Fig. 8).

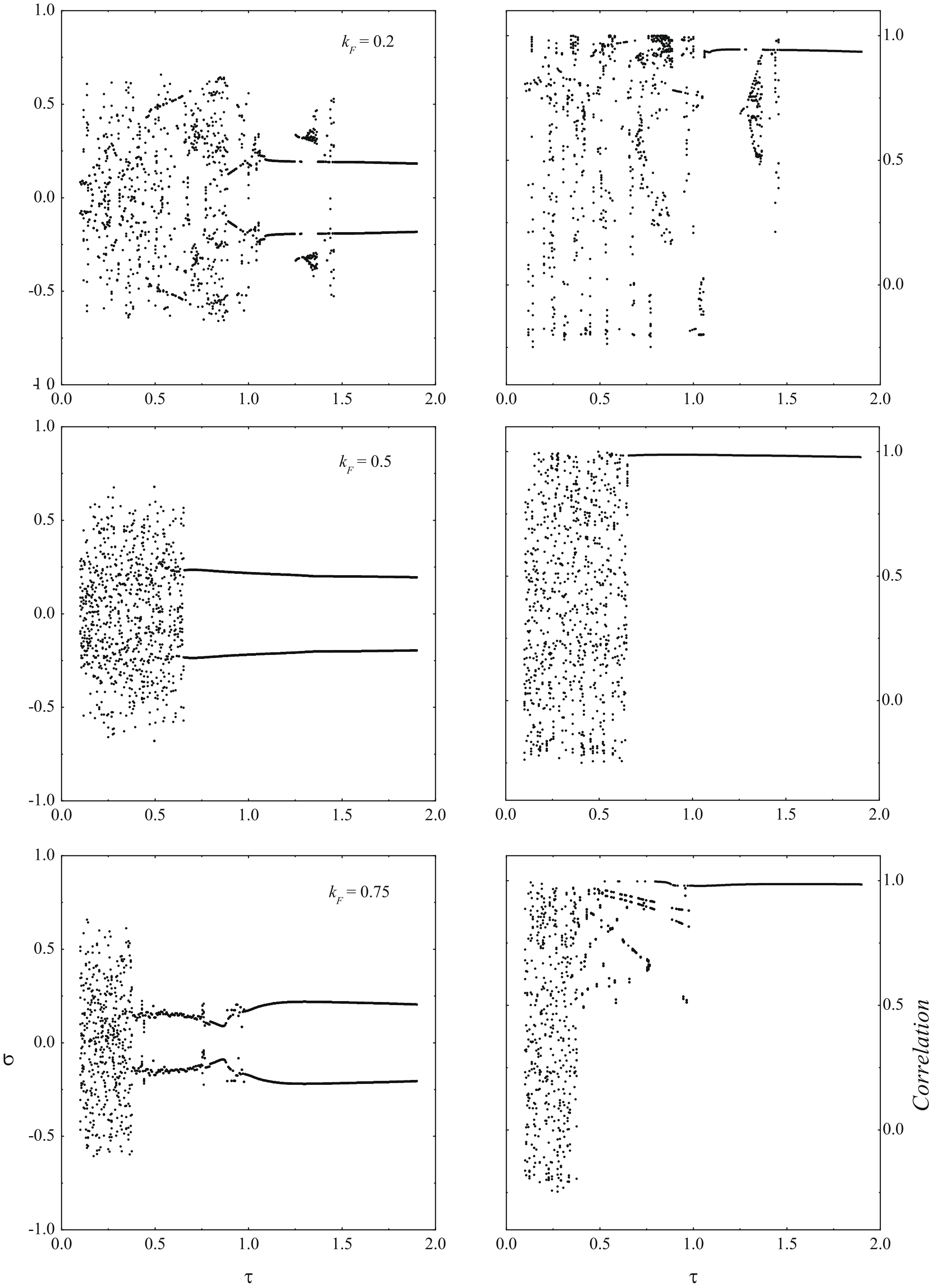

Fig. 8. Bifurcation diagram of the average velocity σ and the correlation function as a function of the coupling period τ for a chain of $N = 5$ pendula, shape parameter $m = 0.999$, and three values of the coupling constant.

3. Conclusion

We have shown that localized reshaping of the driving forces leads to transitions from chaotic to phase-locked behaviour in chains of dissipative and periodically coupled oscillators. In particular, we demonstrated numerically that decreasing the impulse transmitted by the driving pulses of a minimal number of pendula results in regularization with the whole array exhibiting frequency synchronization over a wide range of coupling periods. Also, we showed the great sensitivity of such a regularization scenario against changes in both the coupling period and the coupling constant. Additionally, our results suggest that the present reshaping mechanism can also be employed to enhance chaos in coupled systems [16]. We should stress that this method of regularizing chaotic arrays has potential applications in those cases where the intrinsic parameters of a system cannot be altered while any kind of periodic behaviour is preferred to chaos, such as in superconducting Josephson arrays [17] or semiconductor laser arrays [18].

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